CMP448: Algorithms

Lecture 01: Algorithm Analysis

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Agenda

- Analysis of Algorithms
- Insertion Sort
- Asymptotic Analysis
- Merge Sort
- Recurrences

Acknowledgment
A lot of slides adapted from the slides of Erik Demaine and Charles Leiserson

Algorithms

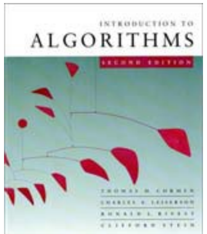
Algorithm

A computational procedure that takes some values as input and produces some values as output

Algorithm Analysis

Determining the *resources* required by the algorithm as a function of the input size.

Resources include *space* and *time*.



Why study algorithms and performance?

- Algorithms help us to understand ***scalability***.
- Performance often draws the line between what is feasible and what is impossible.
- Algorithmic mathematics provides a ***language*** for talking about program behavior.
- Performance is the ***currency*** of computing.
- The lessons of program performance generalize to other computing resources.
- Speed is fun!

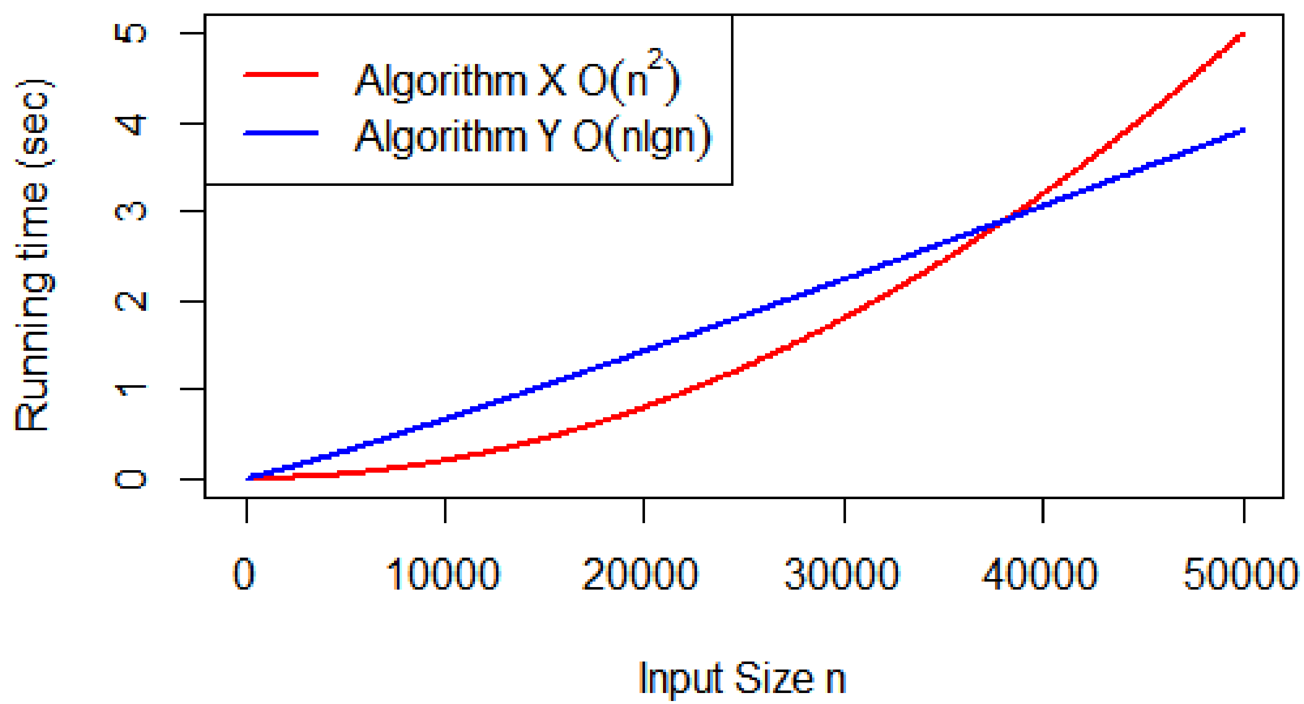
Running time

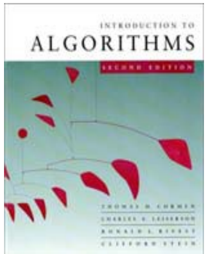
- Assume:
  - Algorithm X takes time $2n^2$ (written by best programmer) running on machine with 1000 MIPS
  - Algorithm Y takes time $50n \lg n$ (written by worst programmer) running on machine with 10 MIPS
- Running time for 10^6 numbers
  - Algorithm X takes 2000 seconds
  - Algorithm Y takes ~100 seconds

Complexity makes a huge difference!

Running time

- Assume:
  - Algorithm X takes time $2n^2$ (written by best programmer) running on machine with 1000 MIPS
  - Algorithm Y takes time $50n \lg n$ (written by worst programmer) running on machine with 10 MIPS





The problem of sorting

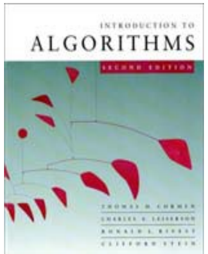
Input: sequence $\langle a_1, a_2, \ldots, a_n \rangle$ of numbers.

Output: permutation $\langle a'_1, a'_2, \ldots, a'_n \rangle$ such that $a'_1 \le a'_2 \le \cdots \le a'_n$.

Example:

Input: 8 2 4 9 3 6

Output: 2 3 4 6 8 9



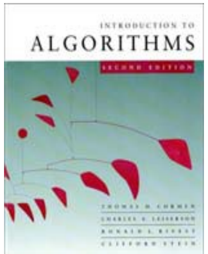
Insertion sort

"pseudocode"

```
INSERTION-SORT (A, n)      ◁ A[1 . . n]
    for j ← 2 to n
        do key ← A[ j]
           i ← j – 1
           while i > 0 and A[i] > key
               do A[i+1] ← A[i]
                  i ← i – 1
           A[i+1] = key
```

Insertion Sort

```
void insertion_sort(vector<int>& A) {
  for (int j = 1; j < A.size(); ++j) {
    int key = A[j];
    int i = j - 1;
    for (; i >= 0 && A[i] > key;) {
      A[i+1] = A[i--];
    }
    A[i+1] = key;
  }
}
```



Insertion sort

"pseudocode"

INSERTION-SORT (A, n) $\triangleleft A[1 \,..\, n]$

    for $j \leftarrow 2$ **to** n

        do $key \leftarrow A[j]$

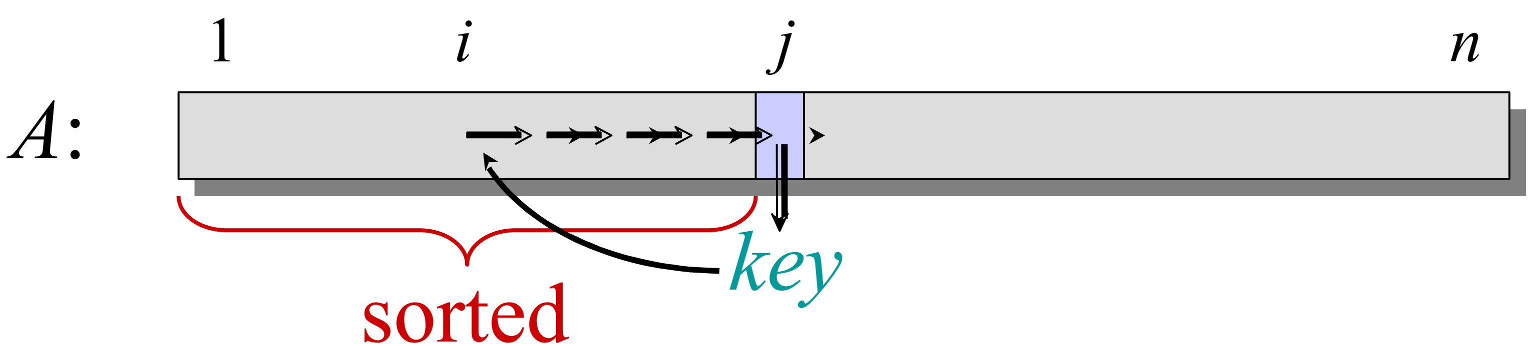
            $i \leftarrow j - 1$

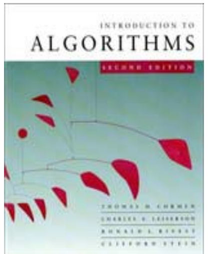
            while $i > 0$ and $A[i] > key$

                do $A[i+1] \leftarrow A[i]$

                    $i \leftarrow i - 1$

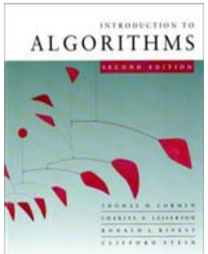
            $A[i+1] = key$





Example of insertion sort

8 2 4 9 3 6



Example of insertion sort

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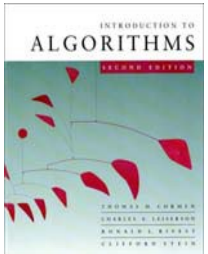

Example of insertion sort

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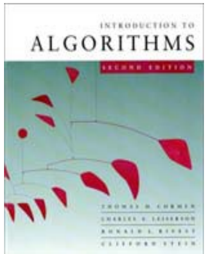
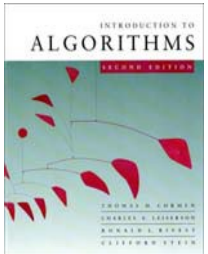


Example of insertion sort

8 **2** 4 9 3 6

2 8 **4** 9 3 6



Example of insertion sort

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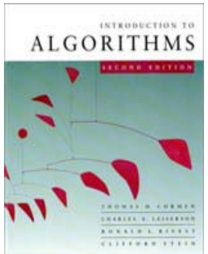
2 8 4 9 3 6

2 4 8 9 3 6



Example of insertion sort

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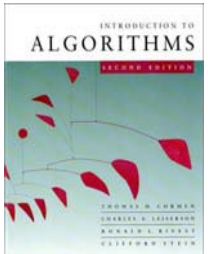


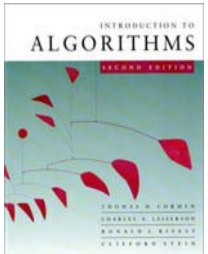
Example of insertion sort

8 **2** 4 9 3 6

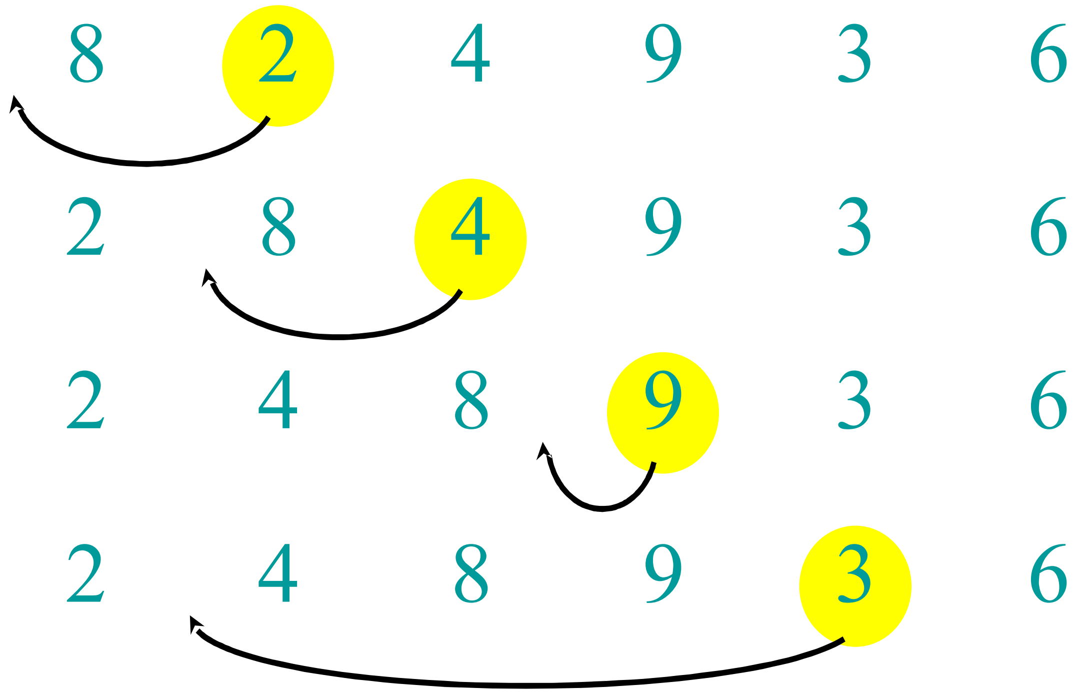
2 8 **4** 9 3 6

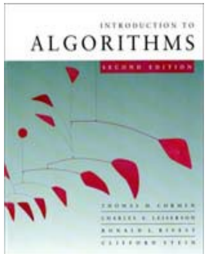
2 4 8 **9** 3 6

2 4 8 9 3 6



Example of insertion sort





Example of insertion sort

8 2 4 9 3 6

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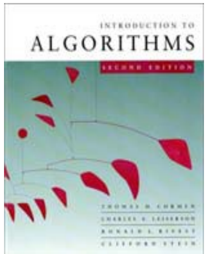
2 3 4 8 9 6



Example of insertion sort

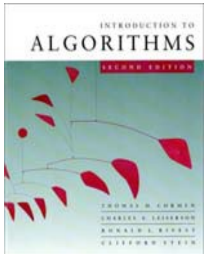
8 **2** 4 9 3 6

2 8 **4** 9 3 6

2 4 8 **9** 3 6

2 4 8 9 **3** 6

2 3 4 8 9 **6**



Example of insertion sort

8 2 4 9 3 6

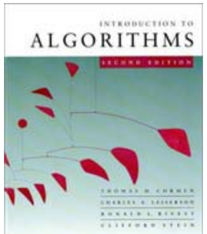
2 8 4 9 3 6

2 4 8 9 3 6

2 4 8 9 3 6

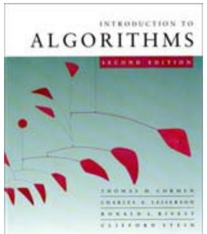
2 3 4 8 9 6

2 3 4 6 8 9 *done*



Running time

- The running time depends on the input: an already sorted sequence is easier to sort.
- Parameterize the running time by the size of the input, since short sequences are easier to sort than long ones.
- Generally, we seek upper bounds on the running time, because everybody likes a guarantee.



Kinds of analyses

Worst-case: (usually)

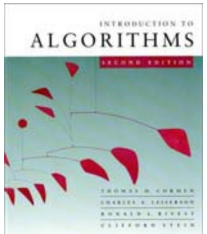
- $T(n)$ = maximum time of algorithm on any input of size n.

Average-case: (sometimes)

- $T(n)$ = expected time of algorithm over all inputs of size n.
- Need assumption of statistical distribution of inputs.

Best-case: (bogus)

- Cheat with a slow algorithm that works fast on *some* input.



Machine-independent time

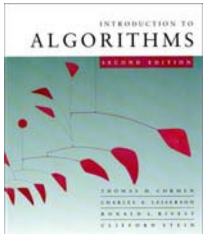
What is insertion sort's worst-case time?

- It depends on the speed of our computer:
    - relative speed (on the same machine),
    - absolute speed (on different machines).

BIG IDEA:

- Ignore machine-dependent constants.
- Look at ***growth*** of $T(n)$ as $n \rightarrow \infty$.

"Asymptotic Analysis"



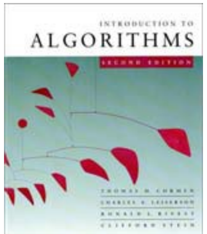
Θ-notation

Math:

$\Theta(g(n)) = \{ f(n)$: there exist positive constants c_1, c_2, and n_0 such that $0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$ for all $n \geq n_0 \}$

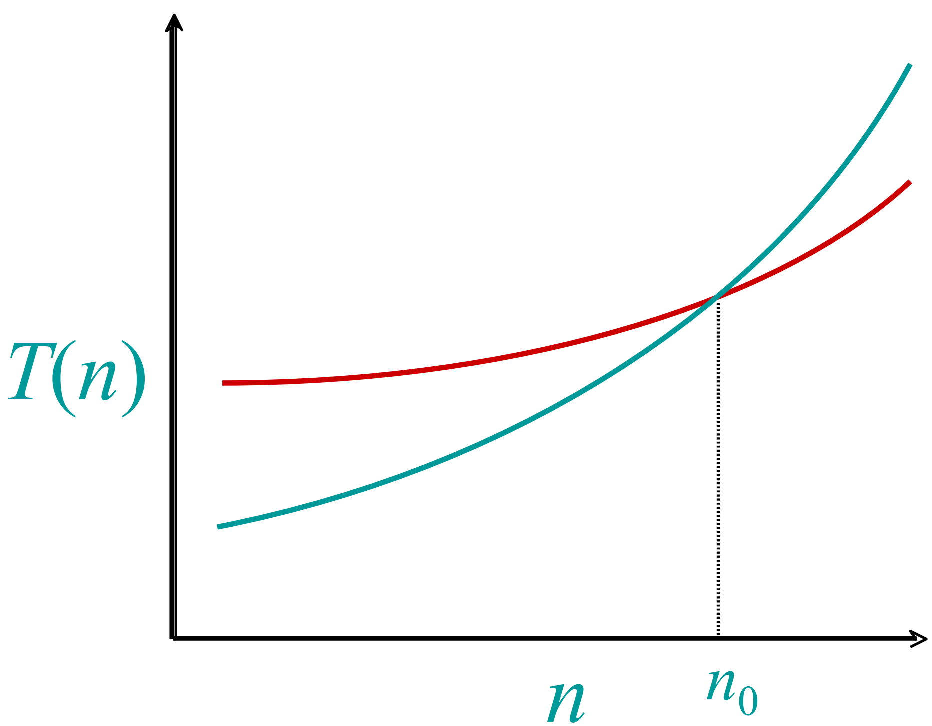
Engineering:

- Drop low-order terms; ignore leading constants.
- Example: $3n^3 + 90n^2 - 5n + 6046 = \Theta(n^3)$



Asymptotic performance

When n gets large enough, a $\Theta(n^2)$ algorithm *always* beats a $\Theta(n^3)$ algorithm.



- We shouldn't ignore asymptotically slower algorithms, however.
- Real-world design situations often call for a careful balancing of engineering objectives.
- Asymptotic analysis is a useful tool to help to structure our thinking.

Insertion Sort Analysis

INSERTION-SORT(*A*)	*cost*	*times*
for $j \leftarrow 2$ **to** $length[A]$	c_1	n
do $key \leftarrow A[j]$	c_2	$n-1$
▷ Insert $A[j]$ into the sorted sequence $A[1\,..\,j-1]$.	0	$n-1$
$i \leftarrow j-1$	c_4	$n-1$
while $i > 0$ and $A[i] > key$	c_5	$\sum_{j=2}^{n} t_j$
do $A[i+1] \leftarrow A[i]$	c_6	$\sum_{j=2}^{n} (t_j - 1)$
$i \leftarrow i-1$	c_7	$\sum_{j=2}^{n} (t_j - 1)$
$A[i+1] \leftarrow key$	c_8	$n-1$

Sum of all terms

$$T(n) = c_1 n + c_2(n-1) + c_4(n-1) + c_5 \sum_{j=2}^{n} t_j + c_6 \sum_{j=2}^{n} (t_j - 1) + c_7 \sum_{j=2}^{n} (t_j - 1) + c_8(n-1)\,.$$

Insertion Sort Analysis

INSERTION-SORT(A)	*cost*	*times*
for $j \leftarrow 2$ **to** *length*[A]	c_1	n
do *key* $\leftarrow A[j]$	c_2	$n-1$
$\triangleright$ Insert $A[j]$ into the sorted sequence $A[1 \,..\, j-1]$.	0	$n-1$
$i \leftarrow j-1$	c_4	$n-1$
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$A[i+1] \leftarrow key$	c_8	$n-1$

Best case: sorted array

$$\sum_{j=2}^{n} t_j = \sum_{j=2}^{n} 1 = n-1$$

$$T(n) = an + b = \Theta(n)$$

Insertion Sort Analysis

INSERTION-SORT(A)	*cost*	*times*
for $j \leftarrow 2$ **to** *length*$[A]$	c_1	n
do *key* $\leftarrow A[j]$	c_2	$n-1$
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$A[i+1] \leftarrow key$	c_8	$n-1$

Worst case: reversed array

$$\sum_{j=2}^{n} t_j = \sum_{j=2}^{n} j = \frac{n(n+1)}{2} - 1$$

Why? $$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$ Why?

Insertion Sort Analysis

$$\sum_{i=1}^{n} i = 1 + \quad 2 + \quad \cdots + \quad n-1 + \quad n$$

$$\sum_{i=1}^{n} i = n + \quad n-1 + \quad \cdots + \quad 2 + \quad 1$$

$$2\sum_{i=1}^{n} i = (n+1) + \quad (n+1) + \quad \cdots + \quad (n+1) + \quad (n+1)$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Review in Appendix A

Insertion Sort Analysis

INSERTION-SORT(A)	*cost*	*times*
for $j \leftarrow 2$ **to** $length[A]$	c_1	n
do $key \leftarrow A[j]$	c_2	$n-1$
▷ Insert $A[j]$ into the sorted sequence $A[1\,..\,j-1]$.	0	$n-1$
$i \leftarrow j-1$	c_4	$n-1$
while $i > 0$ and $A[i] > key$	c_5	$\sum_{j=2}^{n} t_j$
do $A[i+1] \leftarrow A[i]$	c_6	$\sum_{j=2}^{n}(t_j - 1)$
$i \leftarrow i-1$	c_7	$\sum_{j=2}^{n}(t_j - 1)$
$A[i+1] \leftarrow key$	c_8	$n-1$

Worst case: reversed array

$$\sum_{j=2}^{n} t_j = \sum_{j=2}^{n} j = \frac{n(n+1)}{2} - 1$$

$$T(n) = a\,n^2 + bn + c = \Theta(n^2)$$

Insertion Sort Analysis

INSERTION-SORT(A)	*cost*	*times*
for $j \leftarrow 2$ **to** *length*$[A]$	c_1	n
do *key* $\leftarrow A[j]$	c_2	$n-1$
▷ Insert $A[j]$ into the sorted sequence $A[1\,..\,j-1]$.	0	$n-1$
$i \leftarrow j-1$	c_4	$n-1$
while $i > 0$ and $A[i] > key$	c_5	$\sum_{j=2}^{n} t_j$
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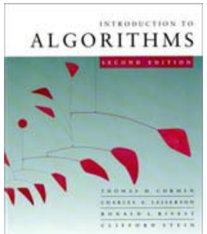
$i \leftarrow i-1$	c_7	$\sum_{j=2}^{n}(t_j-1)$
$A[i+1] \leftarrow key$	c_8	$n-1$

Average case: all permutations equally likely

$$\sum_{j=2}^{n} t_j = \sum_{j=2}^{n} \frac{j}{2}$$

$$T(n) = a\,n^2 + bn + c = \Theta(n^2)$$



Insertion sort analysis

Worst case: Input reverse sorted.

$$T(n) = \sum_{j=2}^{n} \Theta(j) = \Theta(n^2)$$ [arithmetic series]

Average case: All permutations equally likely.

$$T(n) = \sum_{j=2}^{n} \Theta(j/2) = \Theta(n^2)$$

Is insertion sort a fast sorting algorithm?

- Moderately so, for small n.
- Not at all, for large n.

Insertion Sort Analysis

- What about space?
- Insertion sorts "in place" as it does not copy the array anywhere
- It only takes a constant amount of extra storage, independent of n
- Therefore $S(n) = \Theta(1)$

Merge Sort

- Divide-and-Conquer
  - *Divide* the problem into a number of sub-problems
  - *Conquer* the smaller problems
  - *Combine* the results of the sub-problems into a solution for the big problem

MERGE-SORT(A, p, r)
if $p < r$
  then $q \leftarrow \lfloor (p + r)/2 \rfloor$
      MERGE-SORT(A, p, q)
      MERGE-SORT$(A, q + 1, r)$
      MERGE(A, p, q, r)

Merge Sort

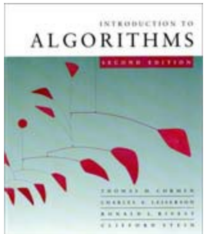
- Divide-and-Conquer
  - *Divide* the problem into a number of sub-problems
  - *Conquer* the smaller problems
  - *Combine* the results of the sub-problems into a solution for the big problem

```
void merge_sort(vector<int>& A, int p, int r) {
  if (p >= r) return;

  int q = (p + r) / 2;

  merge_sort(A, p, q);
  merge_sort(A, q+1, r);

  merge(A, p, q, r);
}
```

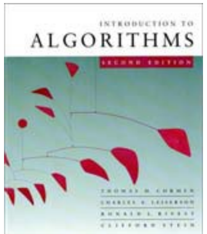


Merge sort

MERGE-SORT $A[1 \,..\, n]$

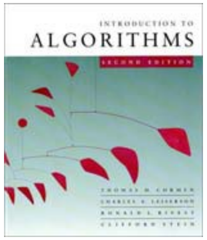
1. If $n = 1$, done.
2. Recursively sort $A[\,1 \,..\, \lceil n/2 \rceil\,]$ and $A[\,\lceil n/2 \rceil + 1 \,..\, n\,]$.
3. "***Merge***" the 2 sorted lists.

Key subroutine: **MERGE**



Merging two sorted arrays

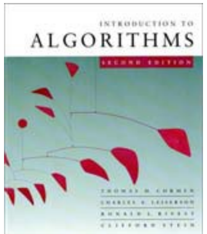
20	12
13	11
7	9
2	1



Merging two sorted arrays

20	12
13	11
7	9
2	1

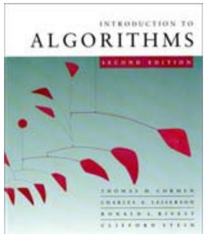
1



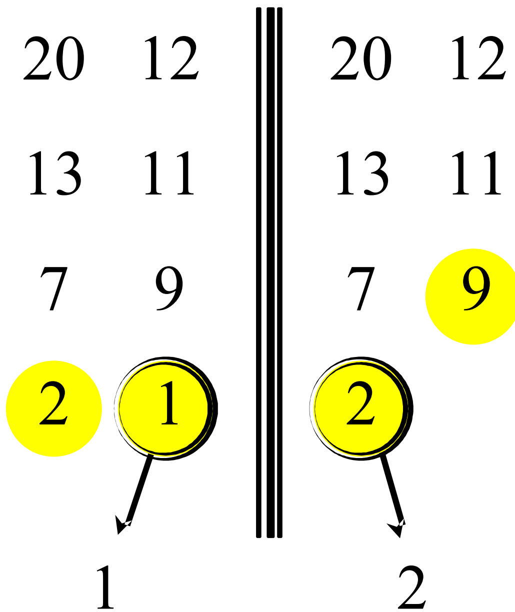
Merging two sorted arrays

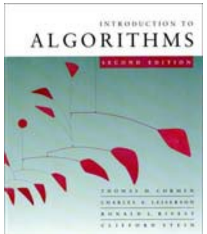
20	12	20	12
13	11	13	11
7	9	7	9
2	1	2	

1

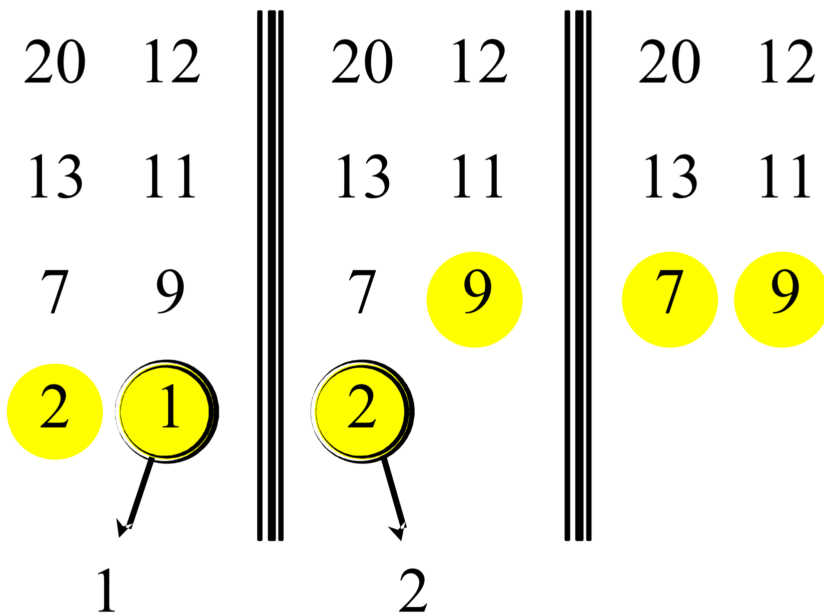


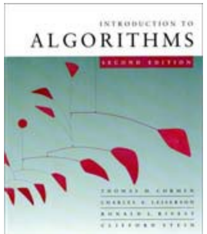
Merging two sorted arrays





Merging two sorted arrays





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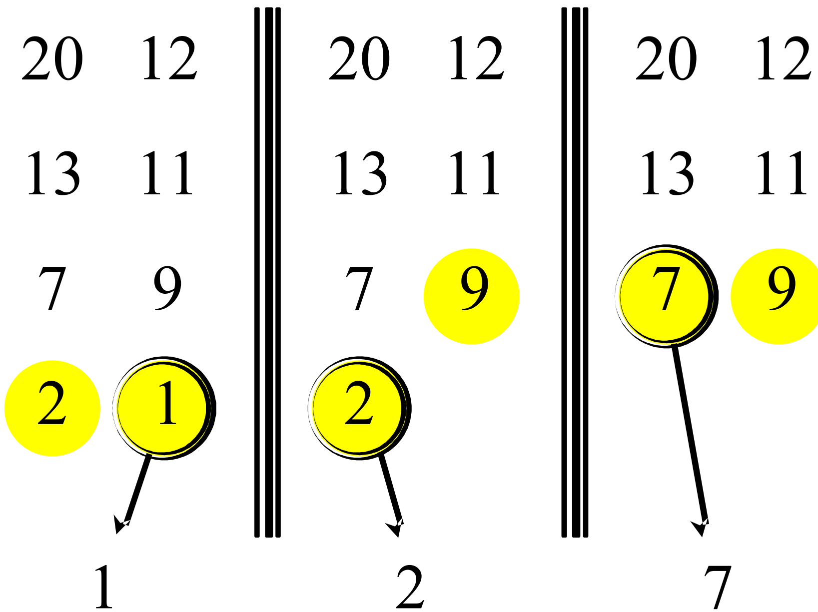
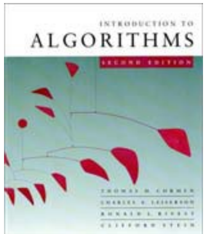
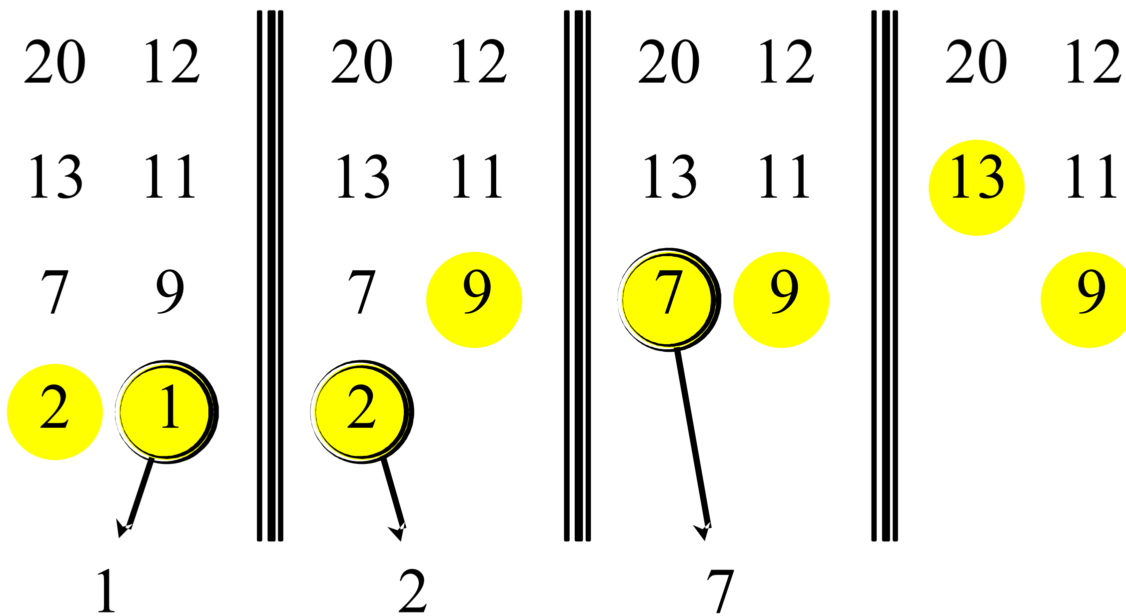
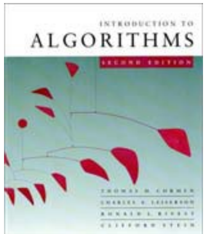




Merging two sorted arrays





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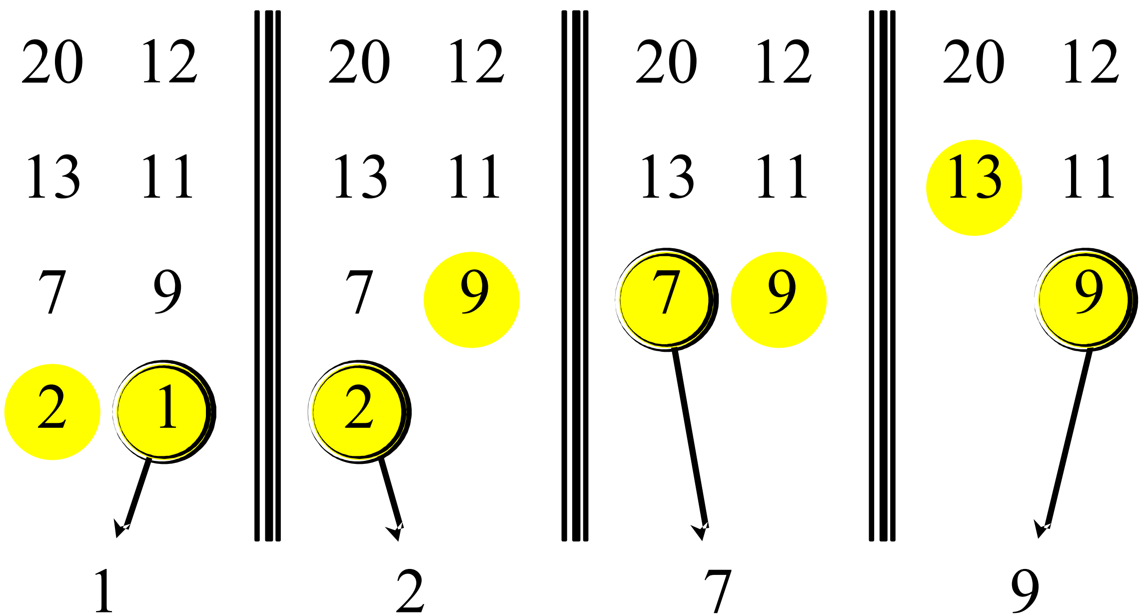

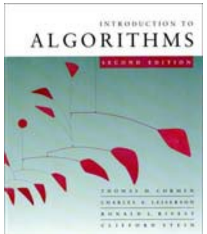

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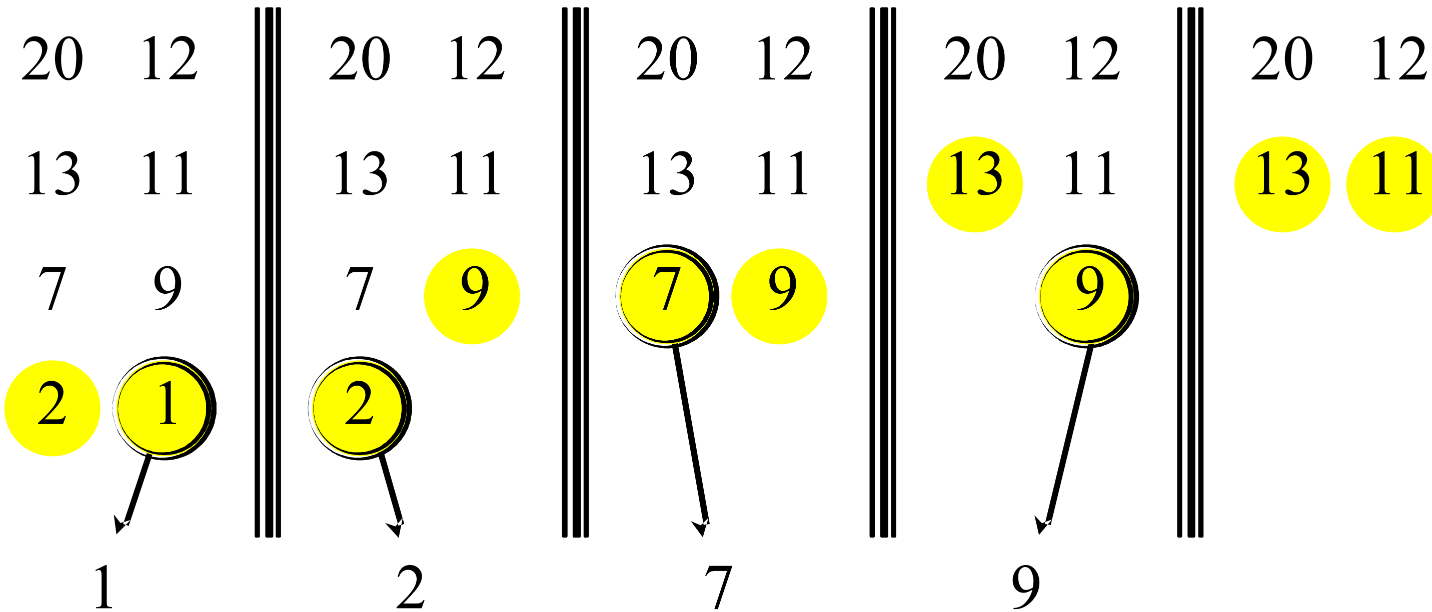

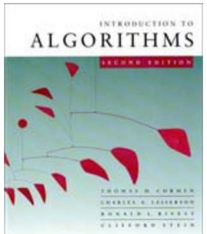

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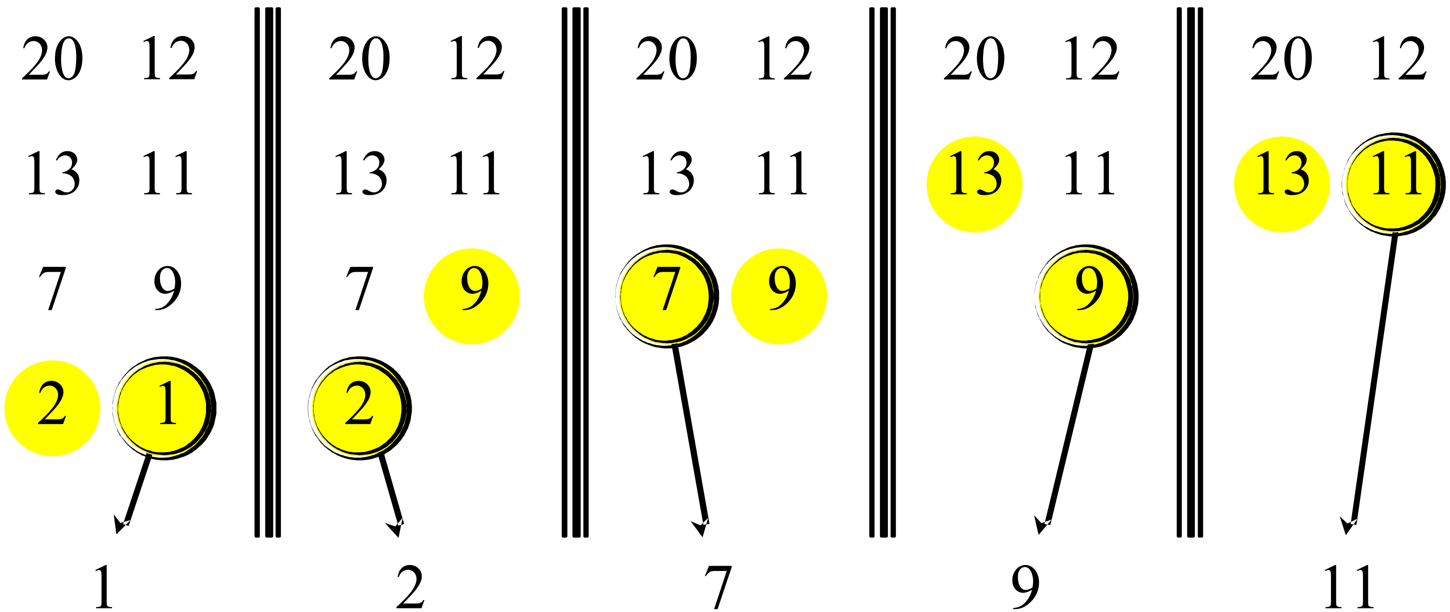

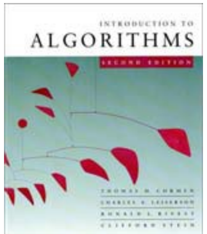

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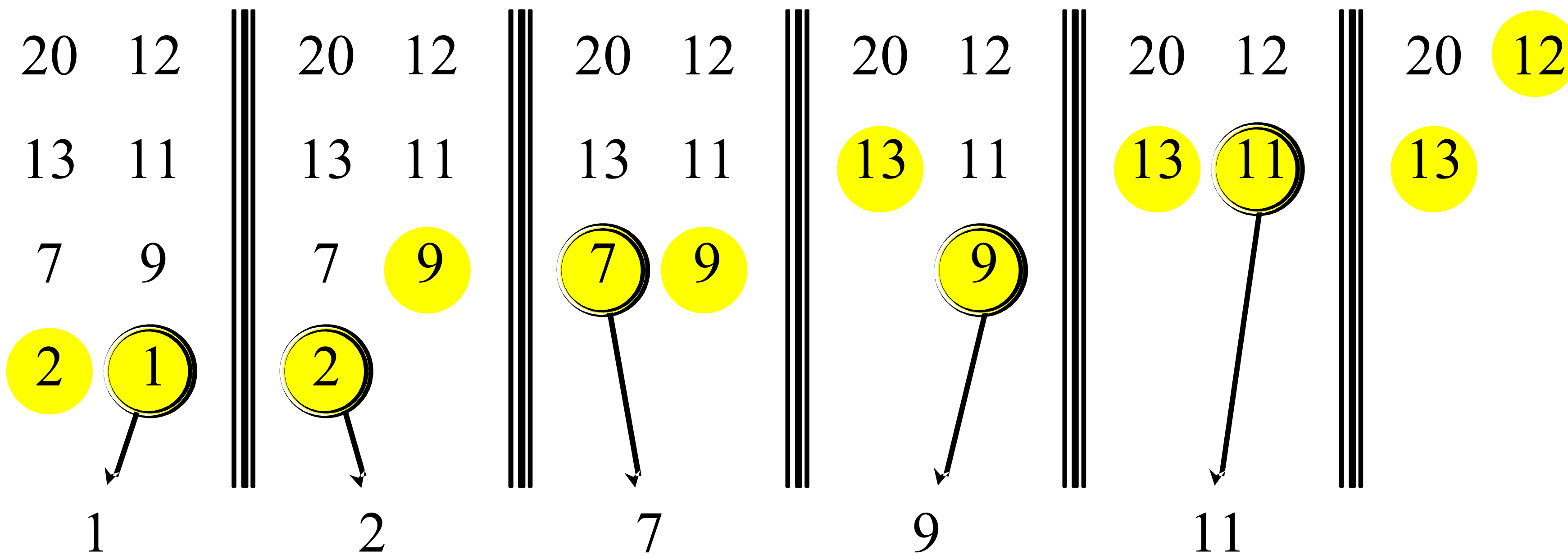

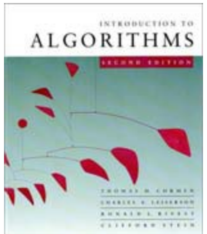

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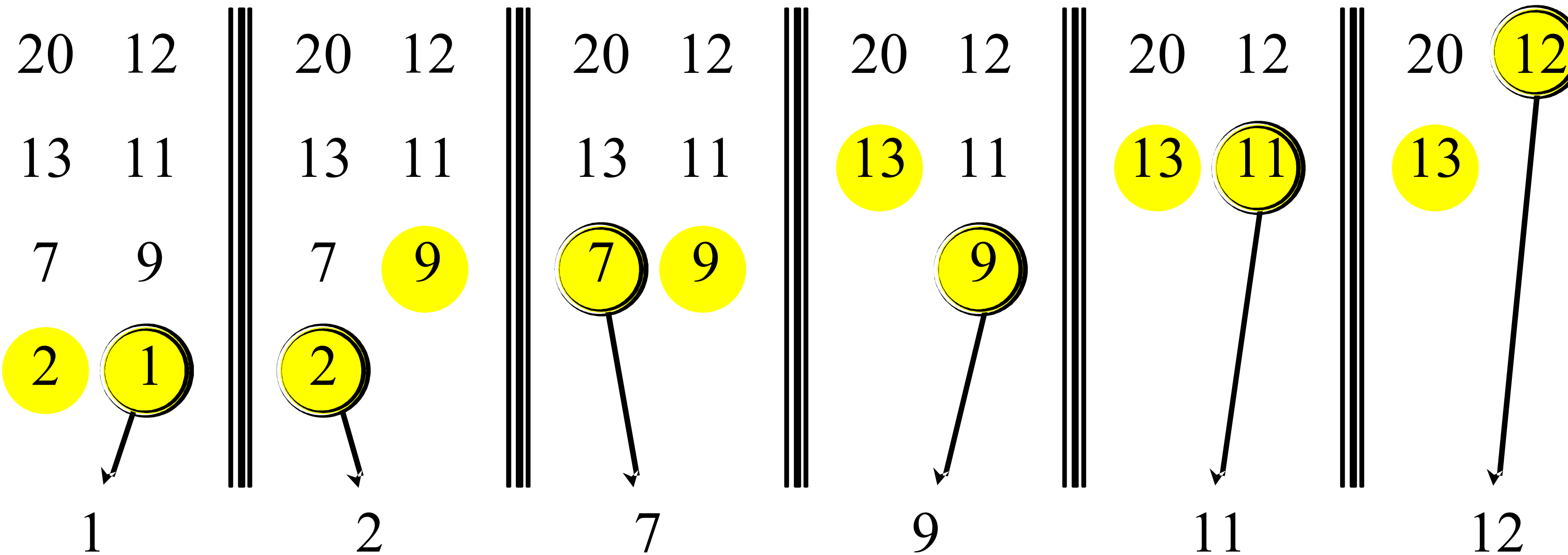

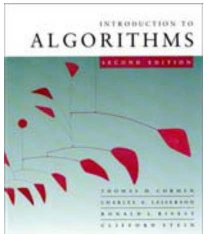

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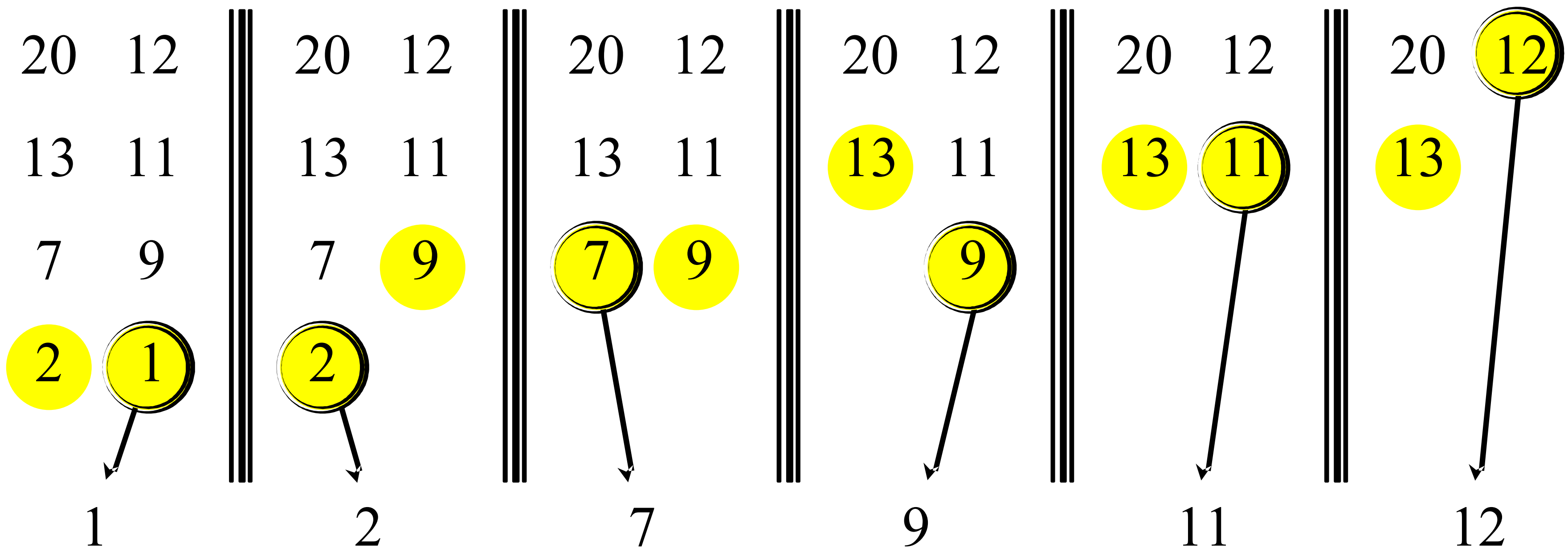


Time = $\Theta(n)$ to merge a total of n elements (linear time).

Merge

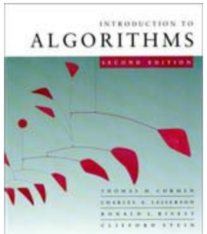
```
MERGE(A, p, q, r)
n1 = q − p + 1
n2 = r − q
let L[1 .. n1 + 1] and R[1 .. n2 + 1] be new arrays
for i = 1 to n1
        L[i] = A[p + i − 1]
for j = 1 to n2
        R[j] = A[q + j]
L[n1 + 1] = ∞
R[n2 + 1] = ∞
i = 1
j = 1
for k = p to r
        if L[i] ≤ R[j]
                A[k] = L[i]
                i = i + 1
        else A[k] = R[j]
                j = j + 1
```

compute sizes

copy arrays to be merged

adds “sentinel”

merge



Analyzing merge sort

$T(n)$	**MERGE-SORT** $A[1 \,.\,.\, n]$
$\Theta(1)$	1. If $n = 1$, done.
$2T(n/2)$	2. Recursively sort $A[\,1 \,.\,.\, \lceil n/2 \rceil\,]$ and $A[\,\lceil n/2 \rceil + 1 \,.\,.\, n\,]$.
$\Theta(n)$	3. ***"Merge"*** the 2 sorted lists

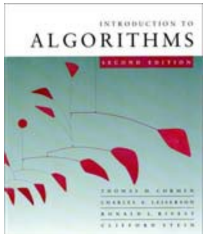
Abuse

Sloppiness: Should be $T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor)$, but it turns out not to matter asymptotically.

Recurrence for Merge Sort

$$T(n) = \begin{cases} \Theta(1) \text{ if } n = 1; \\ 2T(n/2) + \Theta(n) \text{ if } n > 1. \end{cases}$$

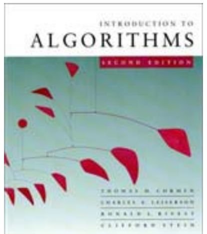
- Next we need to solve this recurrence relation i.e. find *T(n)* as a function of *n* without *T(n/2)*



Recurrence for merge sort

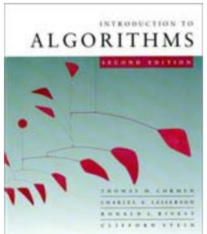
$$T(n) = \begin{cases} \Theta(1) \text{ if } n = 1; \\ 2T(n/2) + \Theta(n) \text{ if } n > 1. \end{cases}$$

- We shall usually omit stating the base case when $T(n) = \Theta(1)$ for sufficiently small n, but only when it has no effect on the asymptotic solution to the recurrence.
- CLRS and Lecture 2 provide several ways to find a good upper bound on $T(n)$.



Recursion tree

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.



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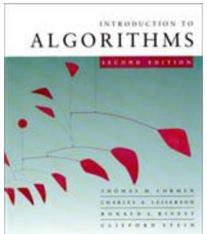
$$T(n)$$



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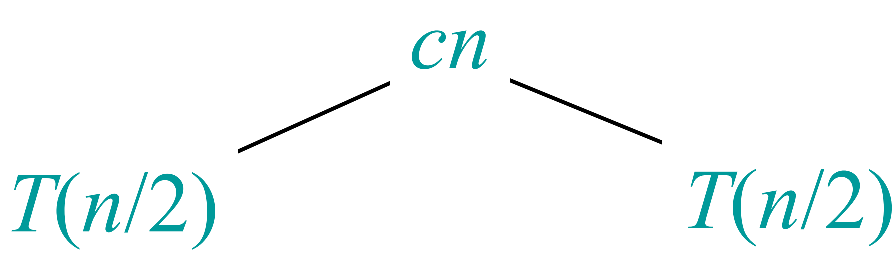

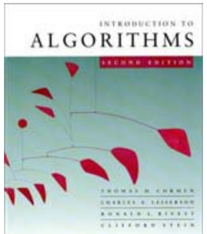

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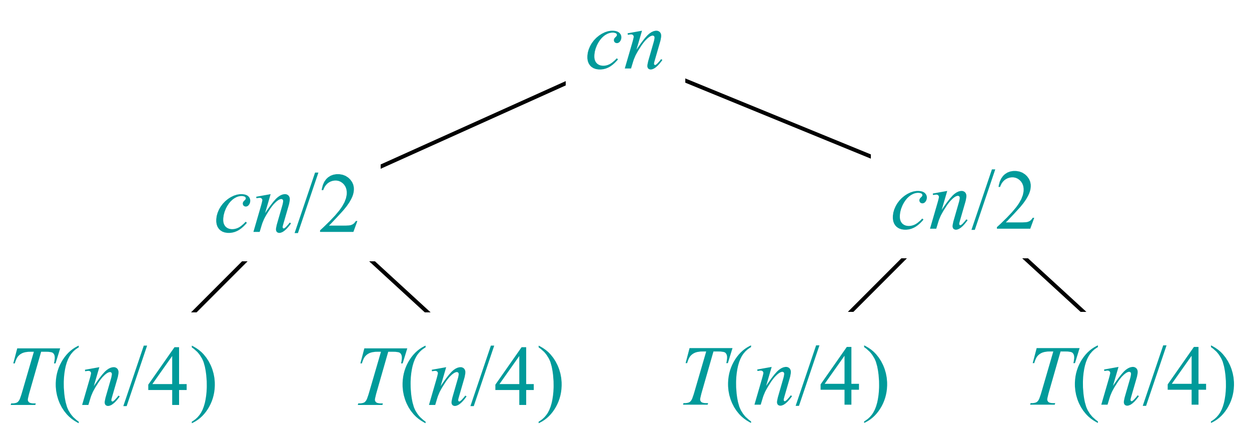

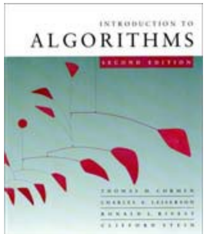

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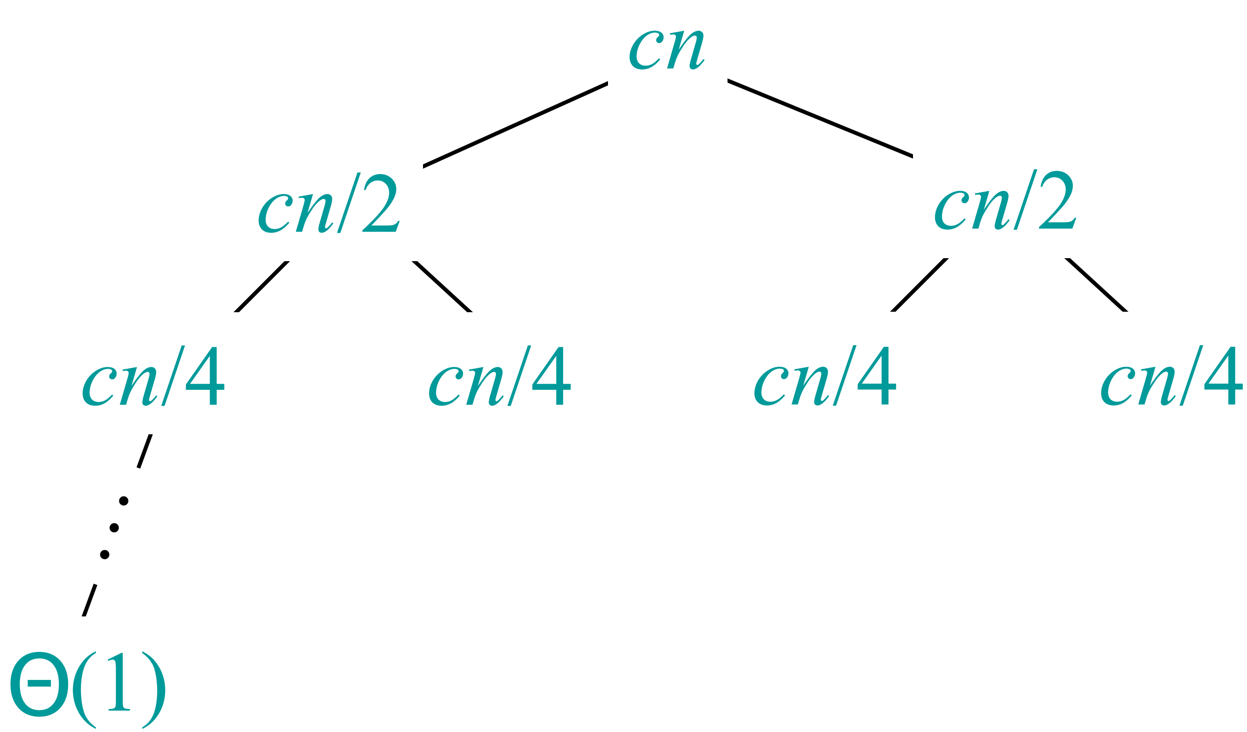

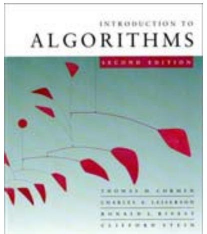

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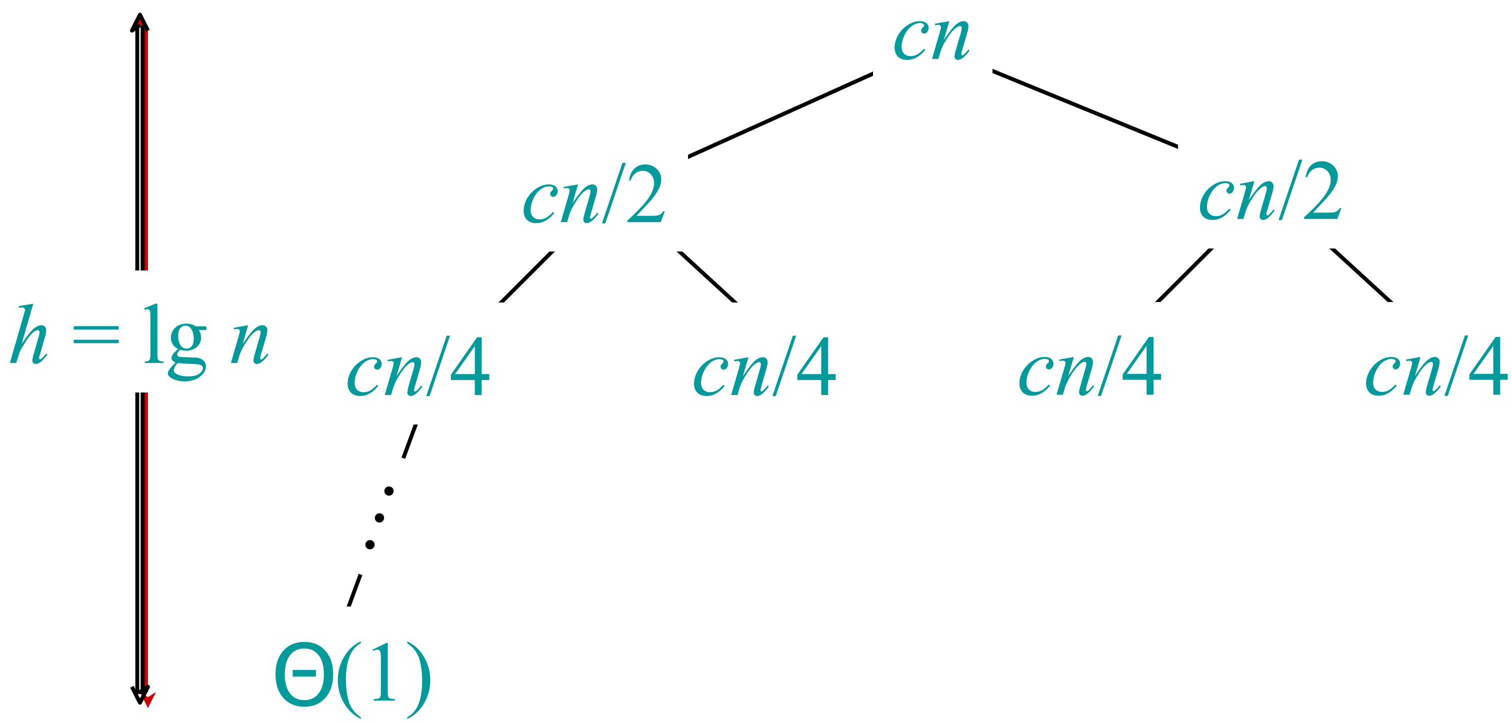

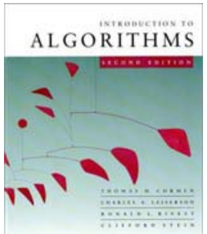

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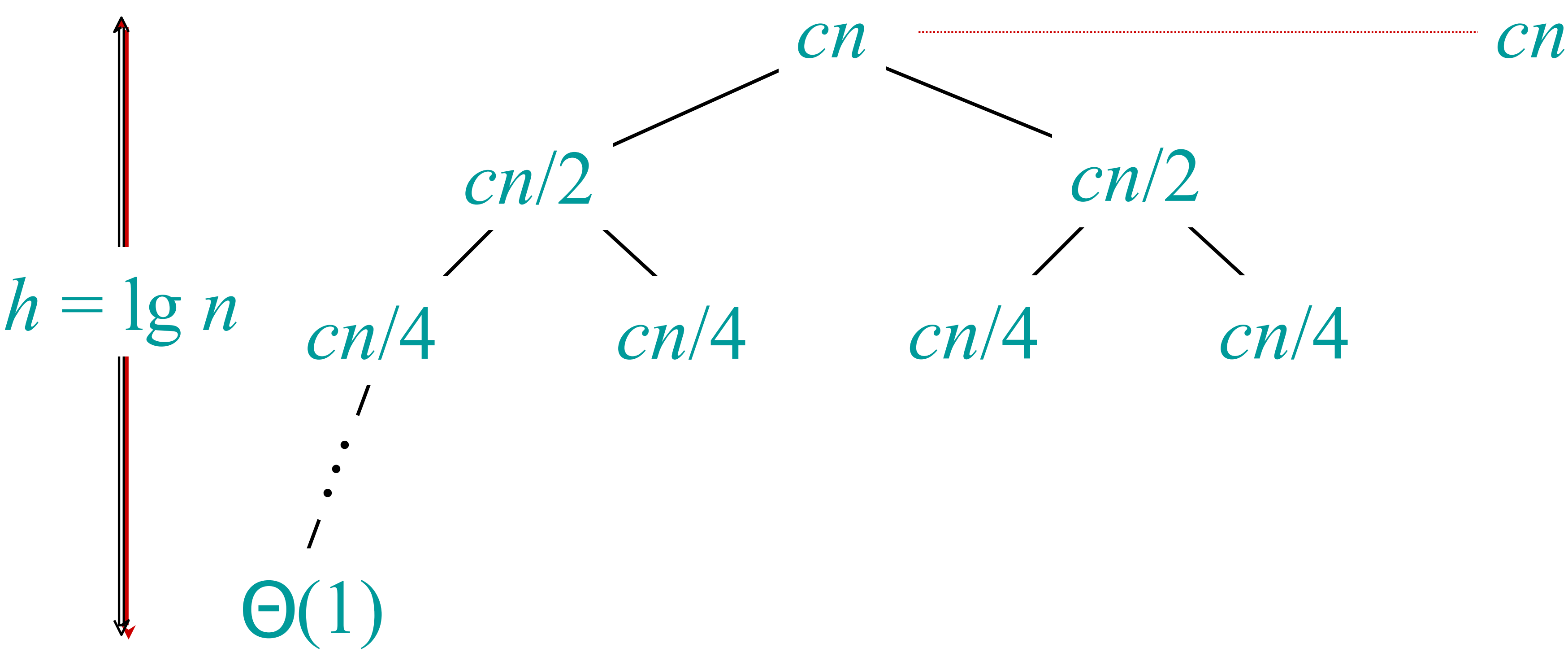

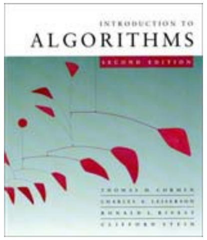

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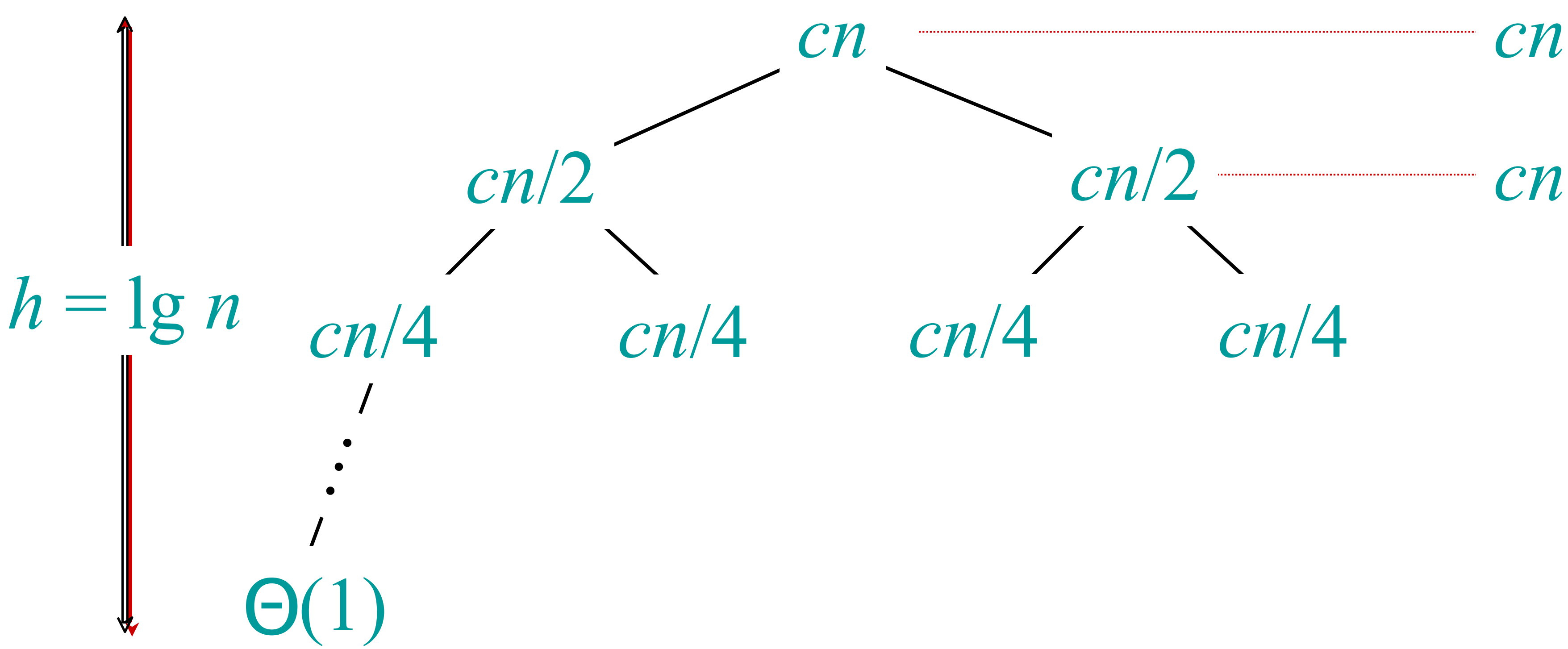

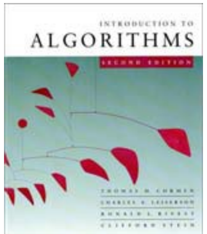

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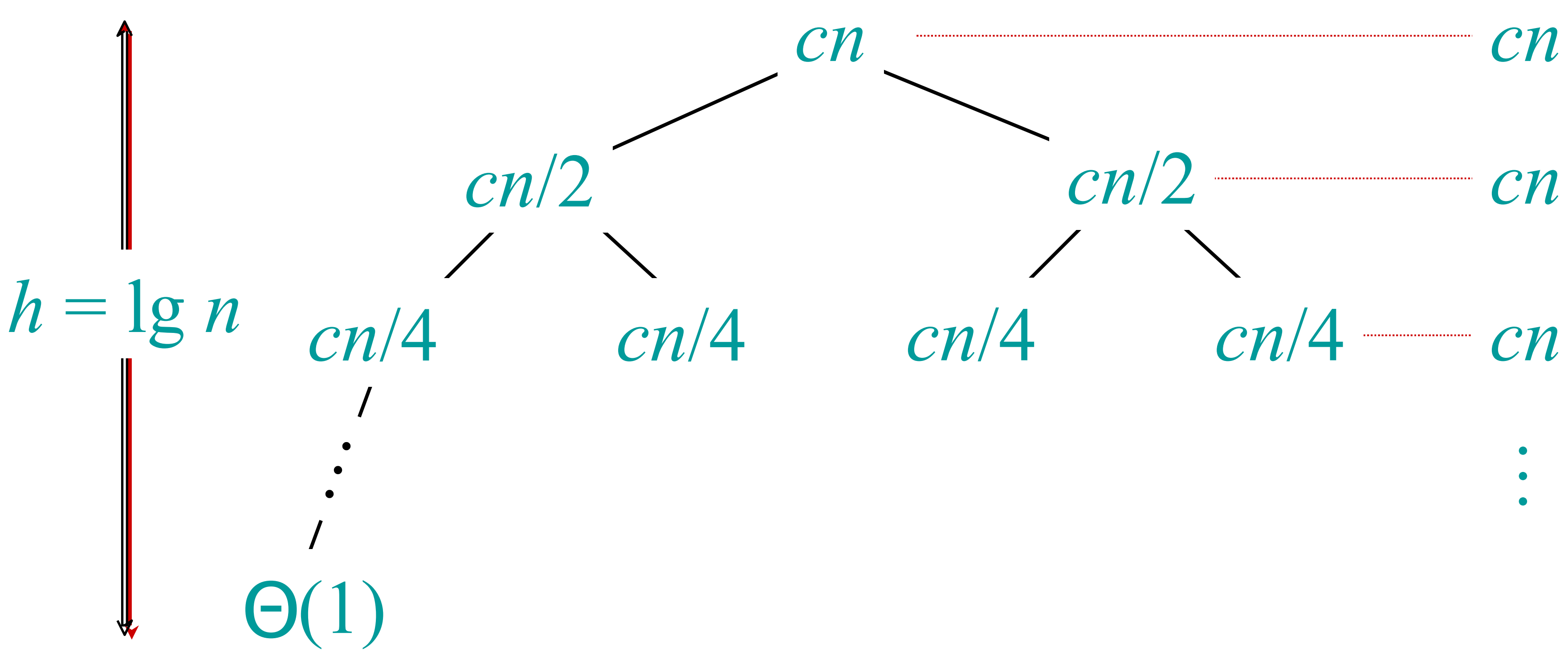

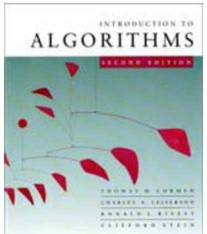

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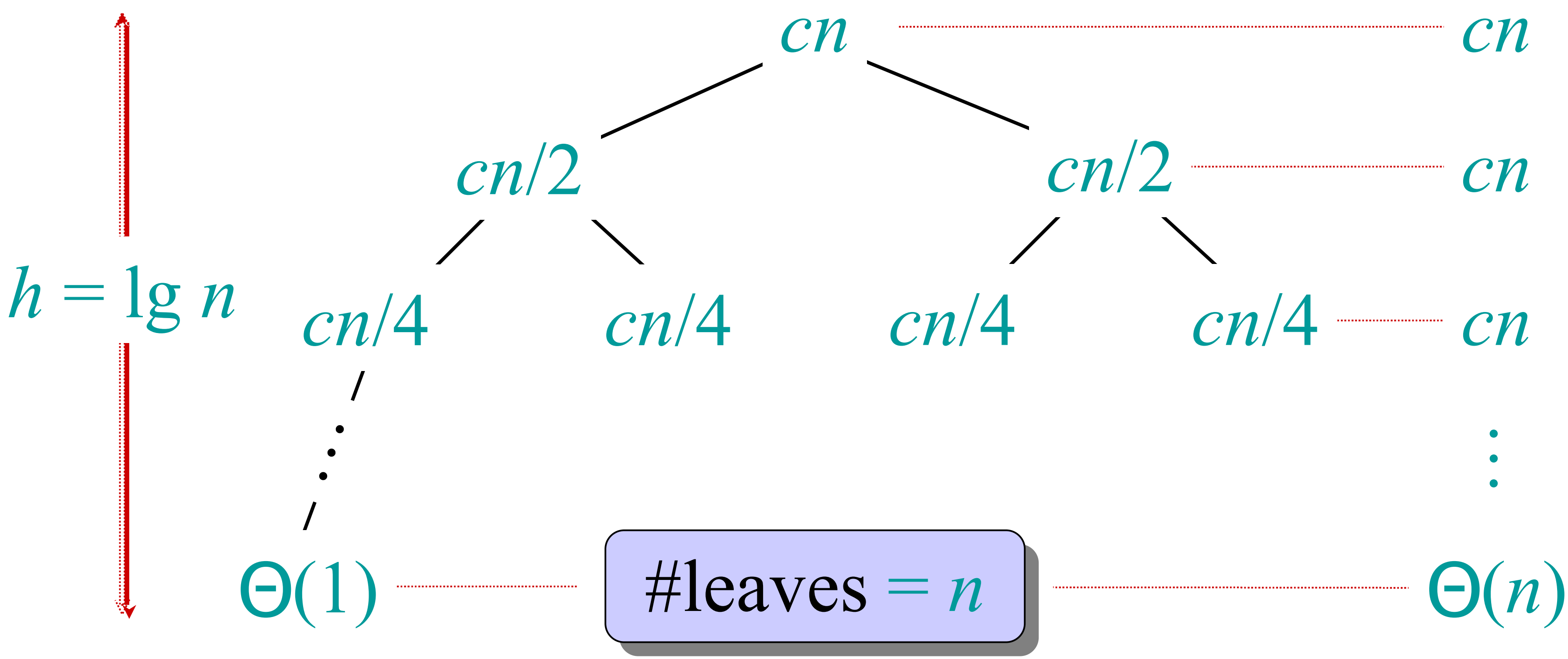

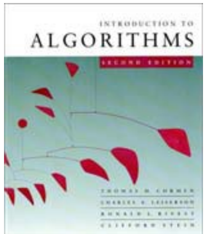

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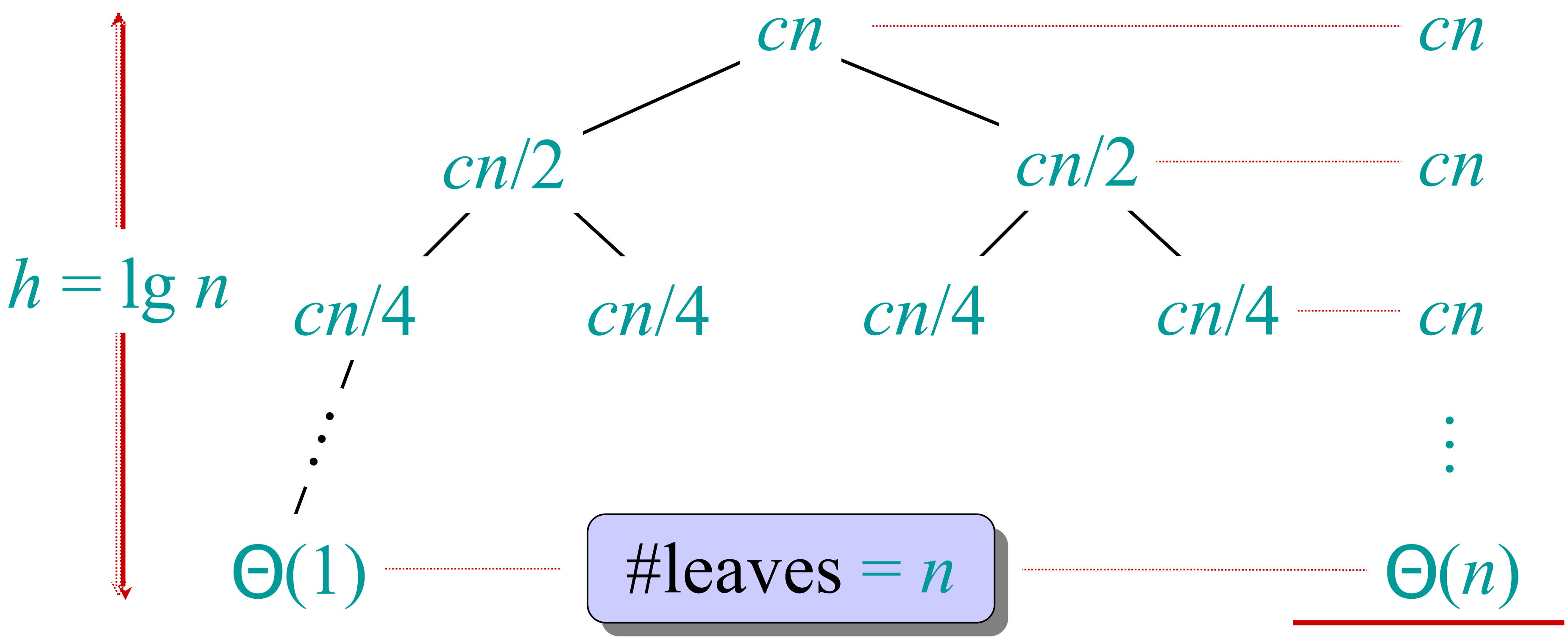


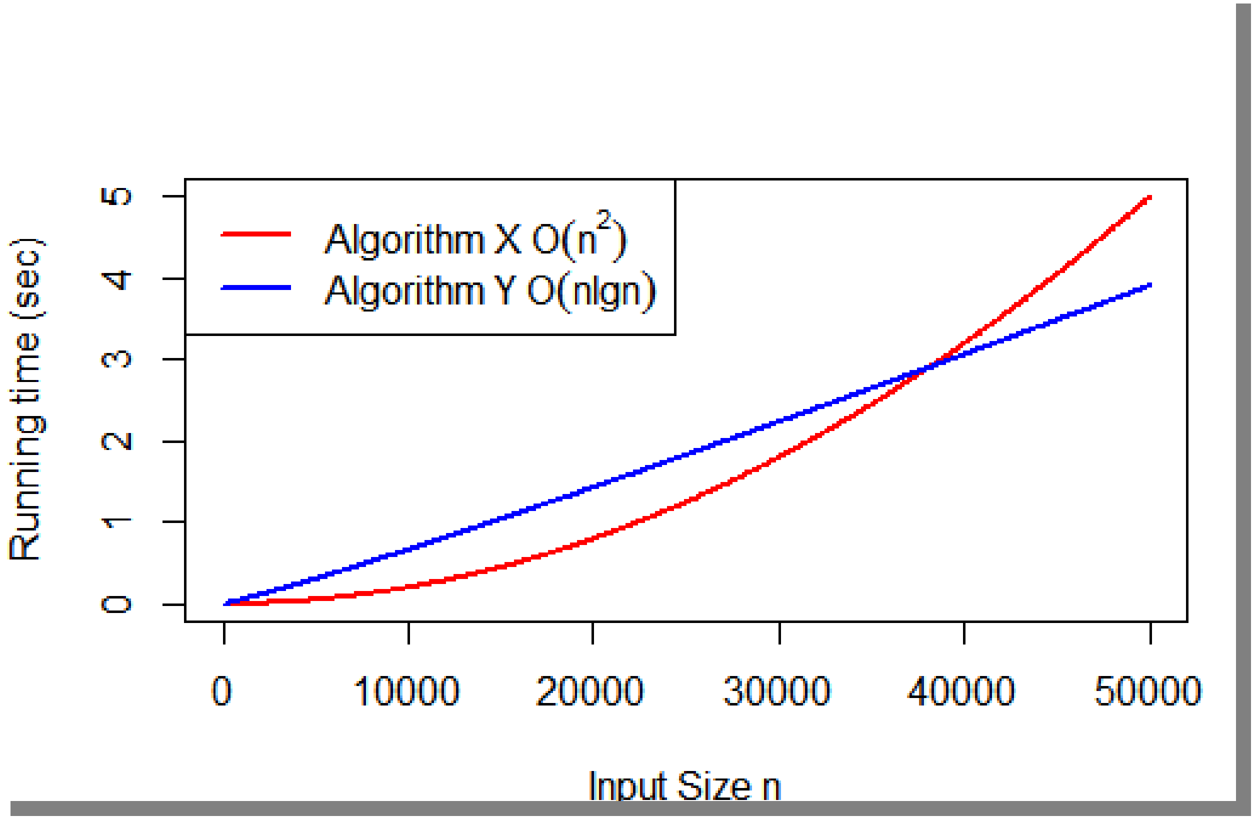
Total $= \Theta(n \lg n)$

Merge Sort Analysis

- What about space?
- Merge Sort does not sort "in place" as it needs temporary space for the "Merge" subroutine
- It needs space: $S(n) = \Theta(n)$

Conclusion

- Insertion Sort:
  - $T(n) = \Theta(n^2)$
  - $S(n) = \Theta(1)$
- Merge Sort:
  - $T(n) = \Theta(n \lg n)$
  - $S(n) = \Theta(n)$



- Insertion sort better for smaller n, merge sort for larger n

Recap

- Insertion Sort
- Merge Sort
- Asymptotic Analysis
- Recurrences
- Next: More on recurrences and asymptotic analysis