CMP448: Algorithms

Lecture 03: Heapsort and Quicksort

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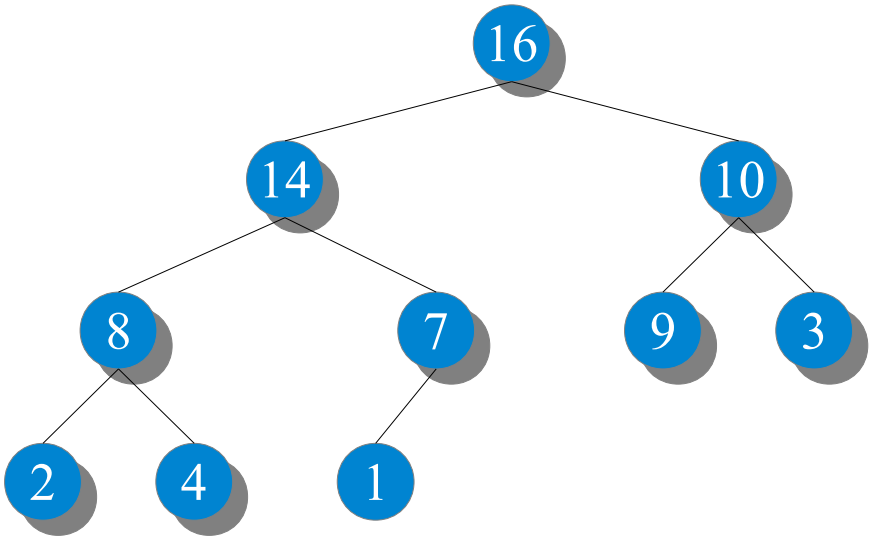
Agenda

- Heapsort
- Quicksort

Acknowledgment
A lot of slides adapted from the slides of Erik Demaine and Charles Leiserson

Heaps

- "Nearly-complete" binary trees i.e. filled except possibly at the leaves
- Each node satisfies the "heap property":
  - Max-Heap: the value of the node ≥ values of the children
  - Min-Heap: the value of the node ≤ values of the children

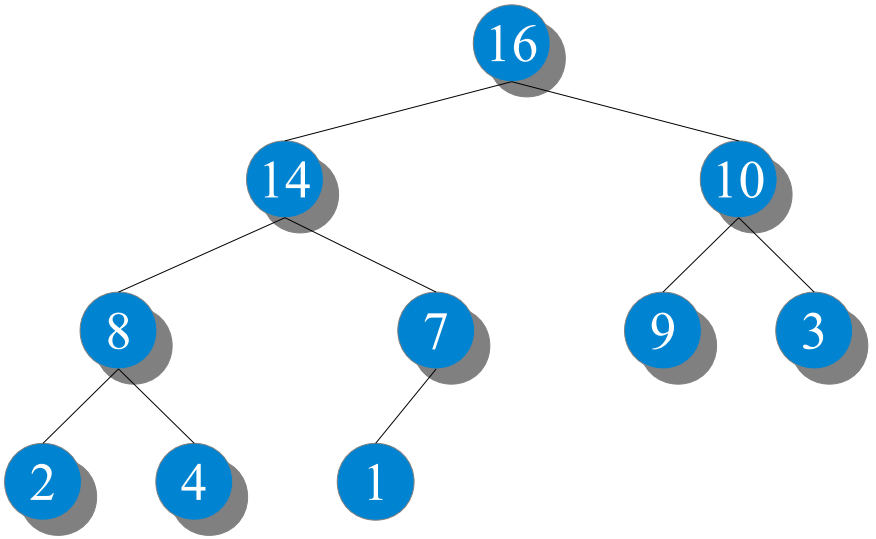


Max-Heap

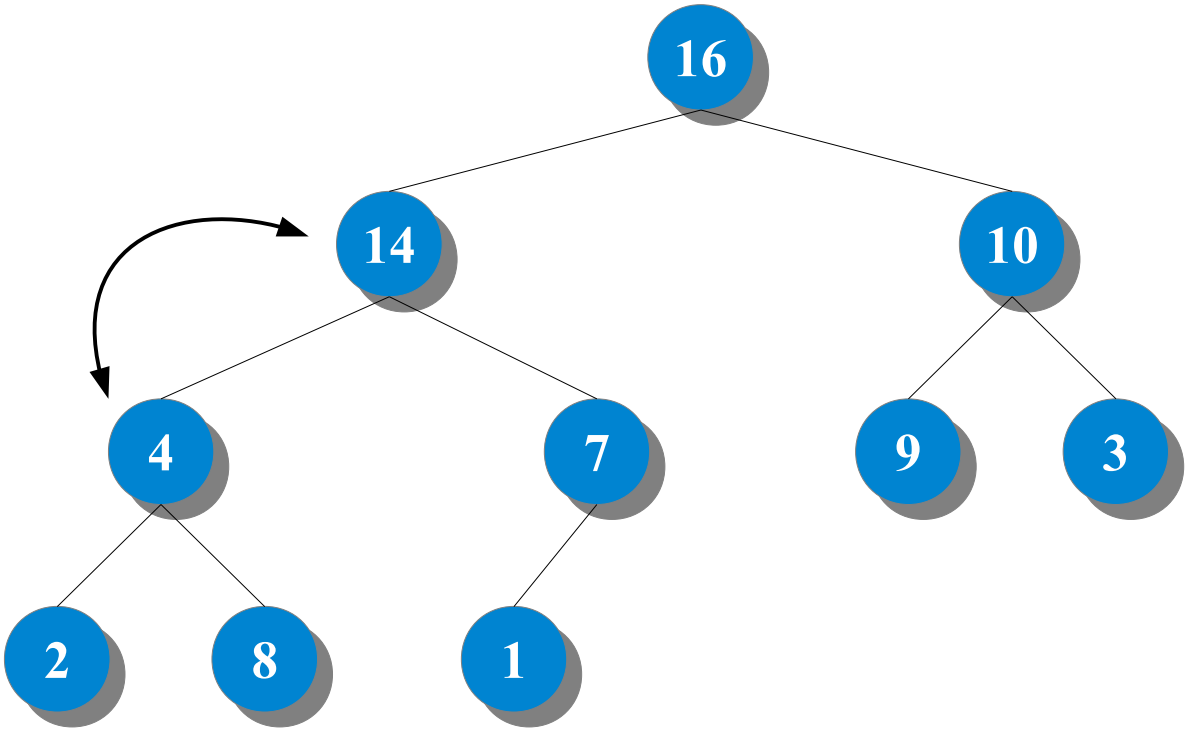
Heaps

- Max-Heap: the value of the node ≥ values of the children
- Fact: The maximum element is at the root.

Max-Heap



Heap Building



1	2	3	4	5	6	7	8	9	10
16	14	10	4	7	9	3	2	8	1

Array Representation

- Heaps are better represented as an array filled from root to leaves
- Three functions required:

Parent of a node i

```
PARENT(i)
1  return ⌊i/2⌋
```

Left child of a node i

```
LEFT(i)
1  return 2i
```

Right child of a node i

```
RIGHT(i)
1  return 2i + 1
```

Properties of array A

A.length: size of array A

A.heap-size: size of heap inside array $A \leq A.length$

Array Representation

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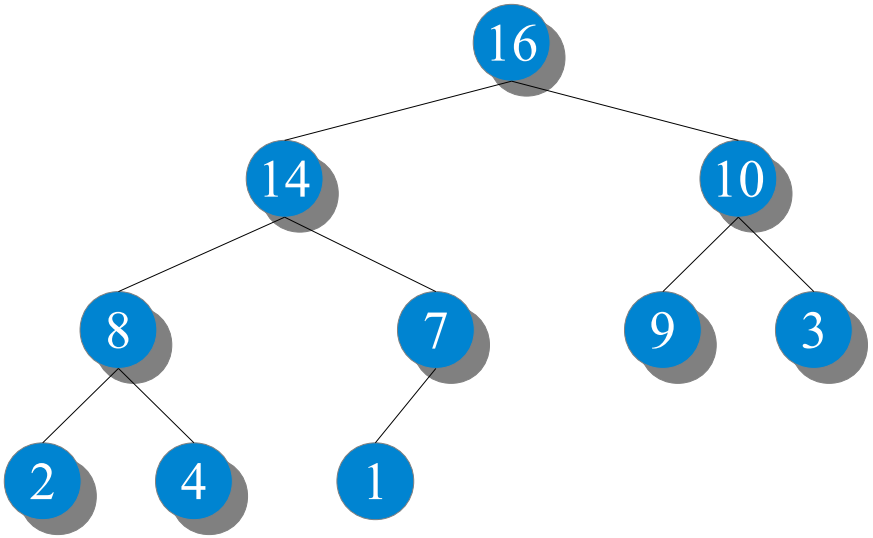

Conceptual Heap Representation

1	2	3	4	5	6	7	8	9	10
16	**14**	**10**	**8**	**7**	**9**	**3**	**2**	**4**	**1**

Array Representation

Array Representation

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1  return ⌊i/2⌋
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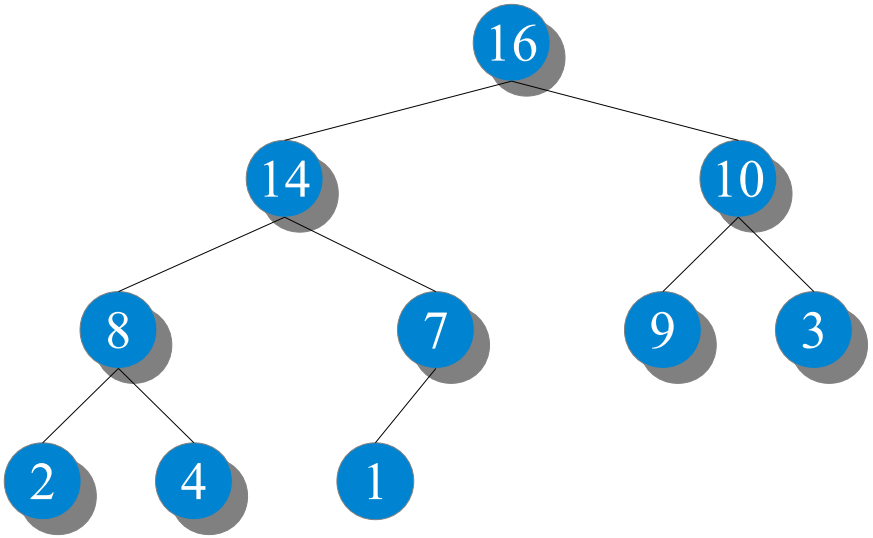
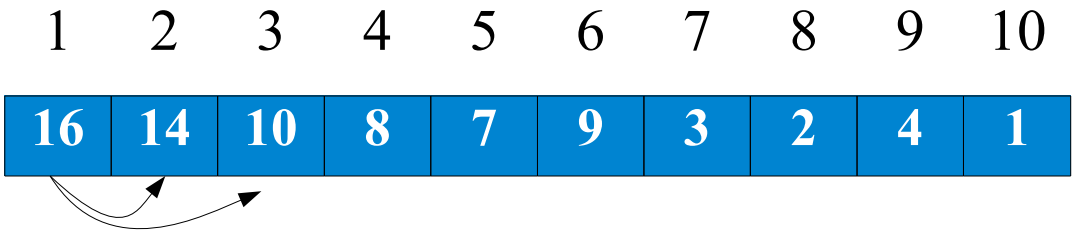


Conceptual Heap Representation



Array Representation

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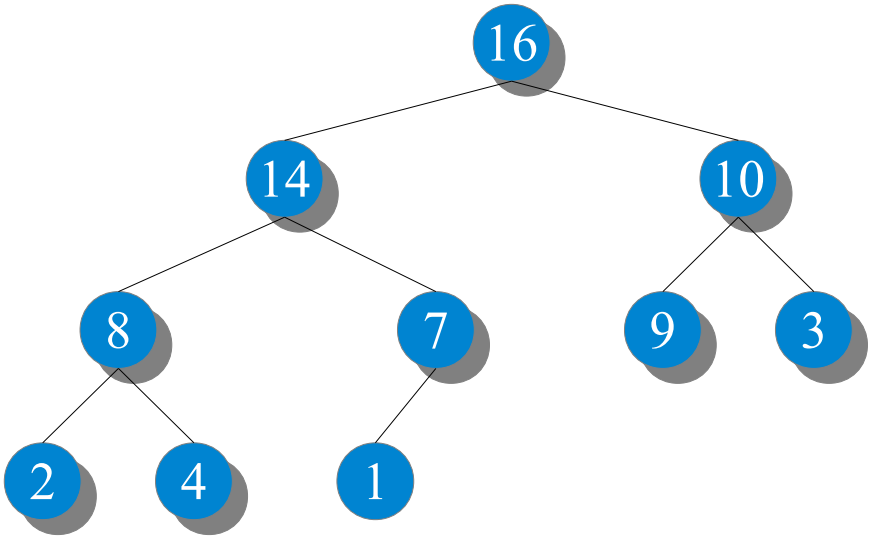
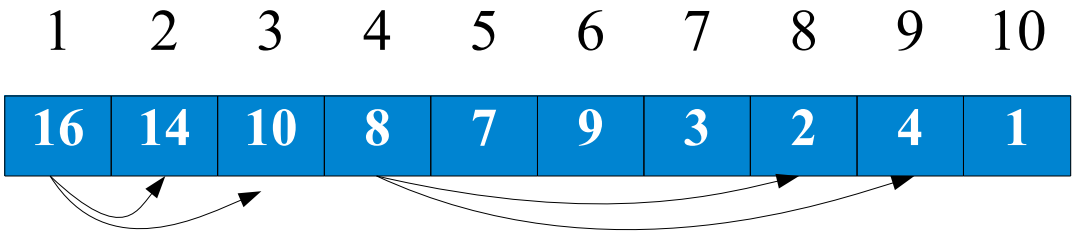


Conceptual Heap Representation

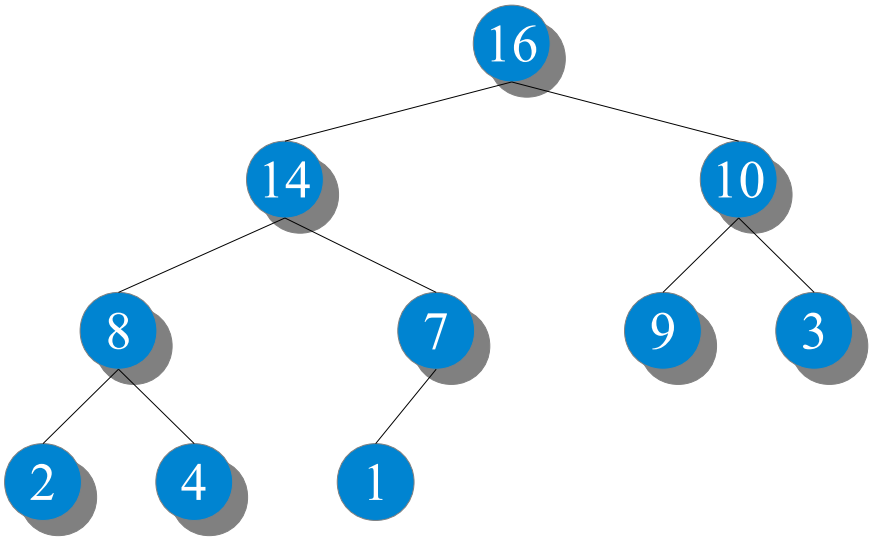


Array Representation

Max-Heaps

Max-Heap Property

$$A[Parent(i)] \geq A[i]$$



Properties of array *A*

A.length: size of array *A*

A.heap-size: size of heap inside array $A \leq A.length$

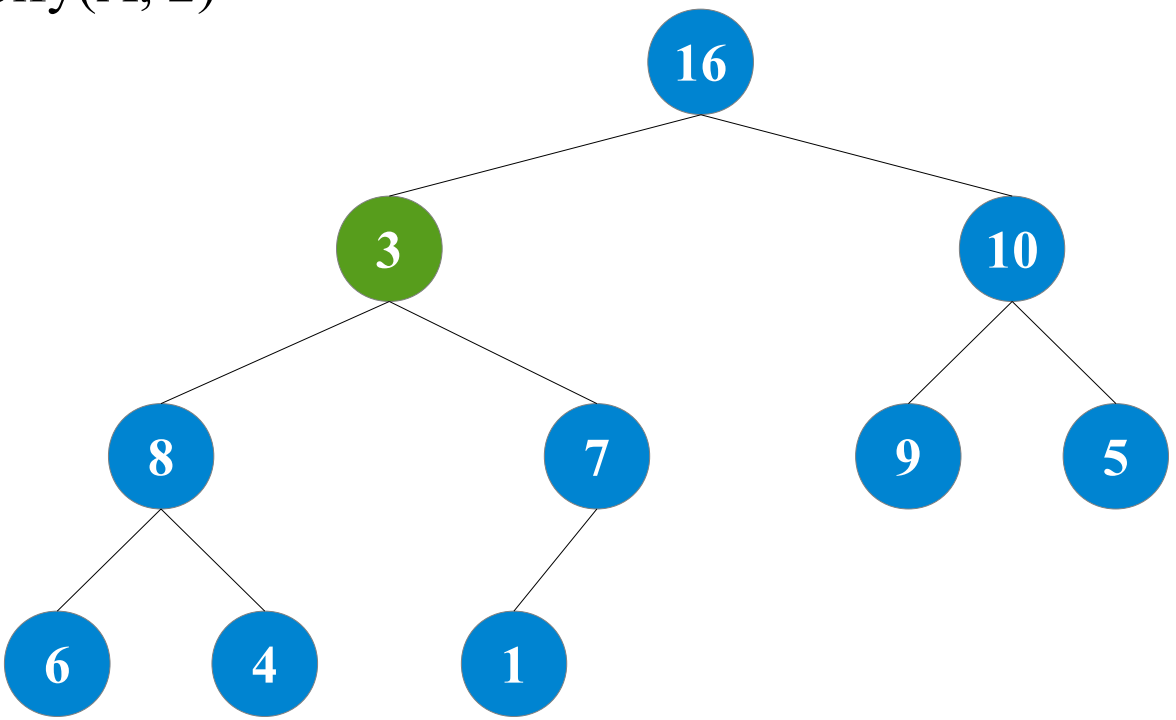
Max-Heapify

- Maintains the Max-Heap property of the heap
- Starting from a node in the heap that might violate the heap property i.e. smaller than its children

```
MAX-HEAPIFY (A, i)
1 l = LEFT(i)
2 r = RIGHT(i)
3 if l ≤ A.heap-size and A[l] > A[i]
4     largest = l
5 else largest = i
6 if r ≤ A.heap-size and A[r] > A[largest]
7     largest = r
8 if largest ≠ i
9     exchange A[i] with A[largest]
10    MAX-HEAPIFY(A, largest)
```

Max-Heapify

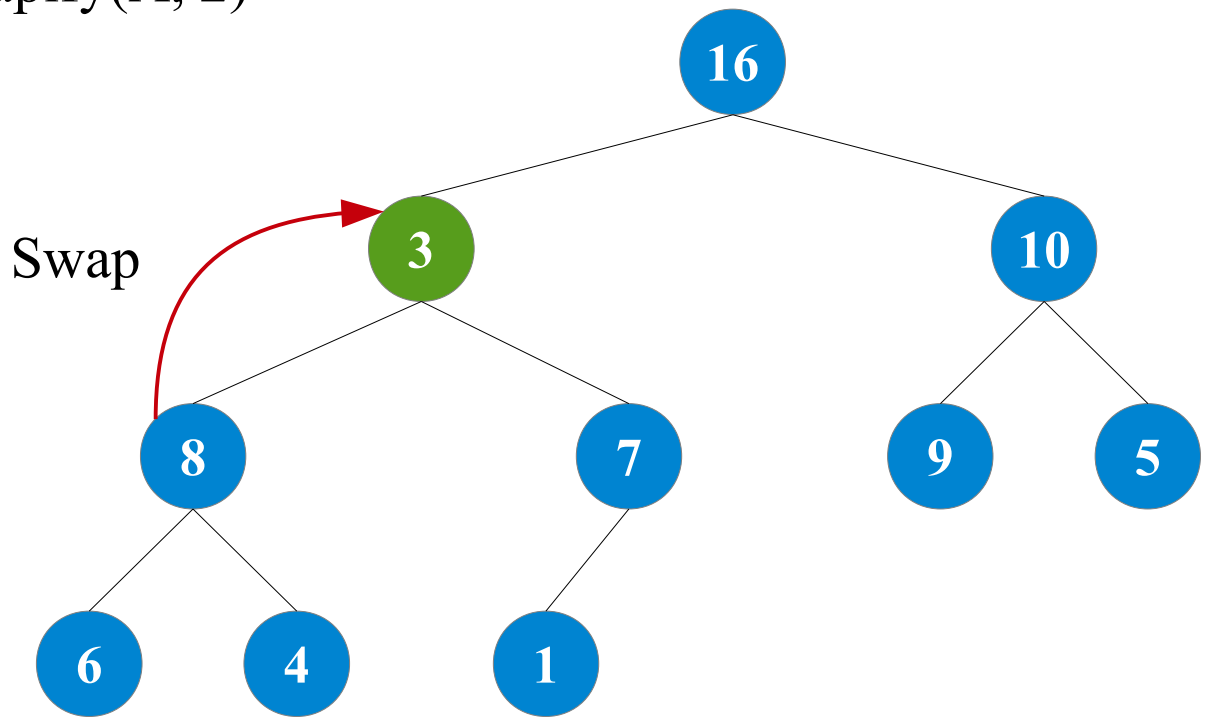
Max-Heapify(A, 2)



1	2	3	4	5	6	7	8	9	10
16	3	10	8	7	9	5	6	4	1

Max-Heapify

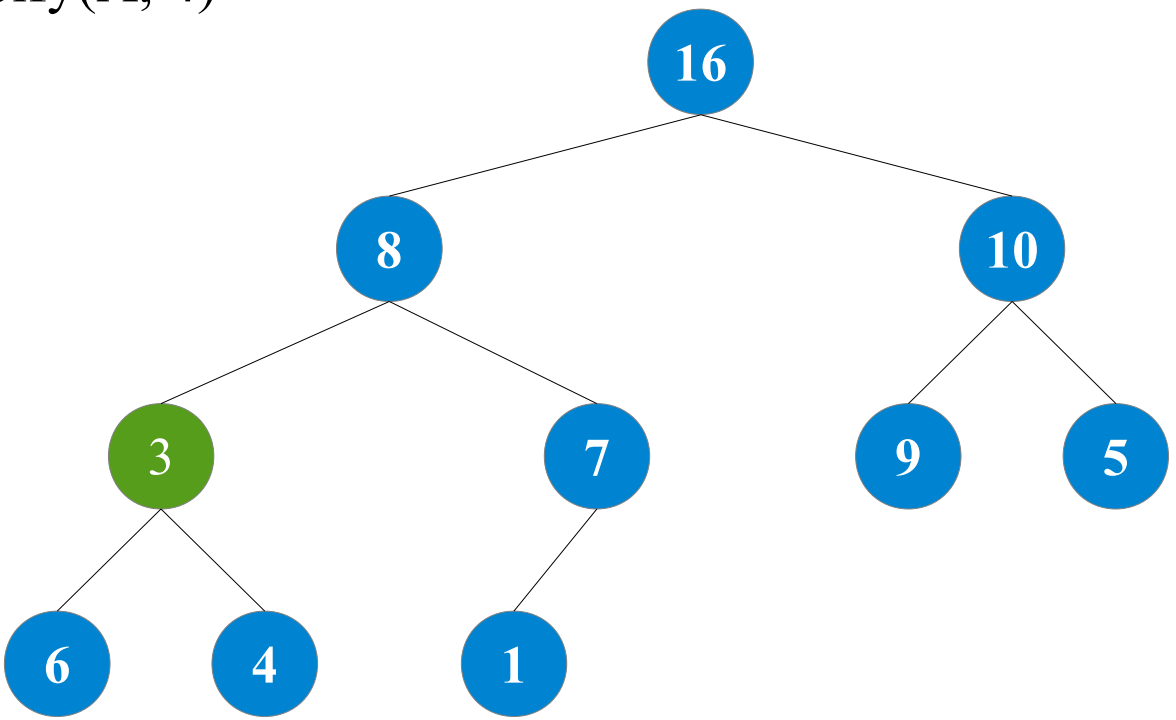
Max-Heapify(A, 2)



1	2	3	4	5	6	7	8	9	10
16	3	10	8	7	9	5	6	4	1

Max-Heapify

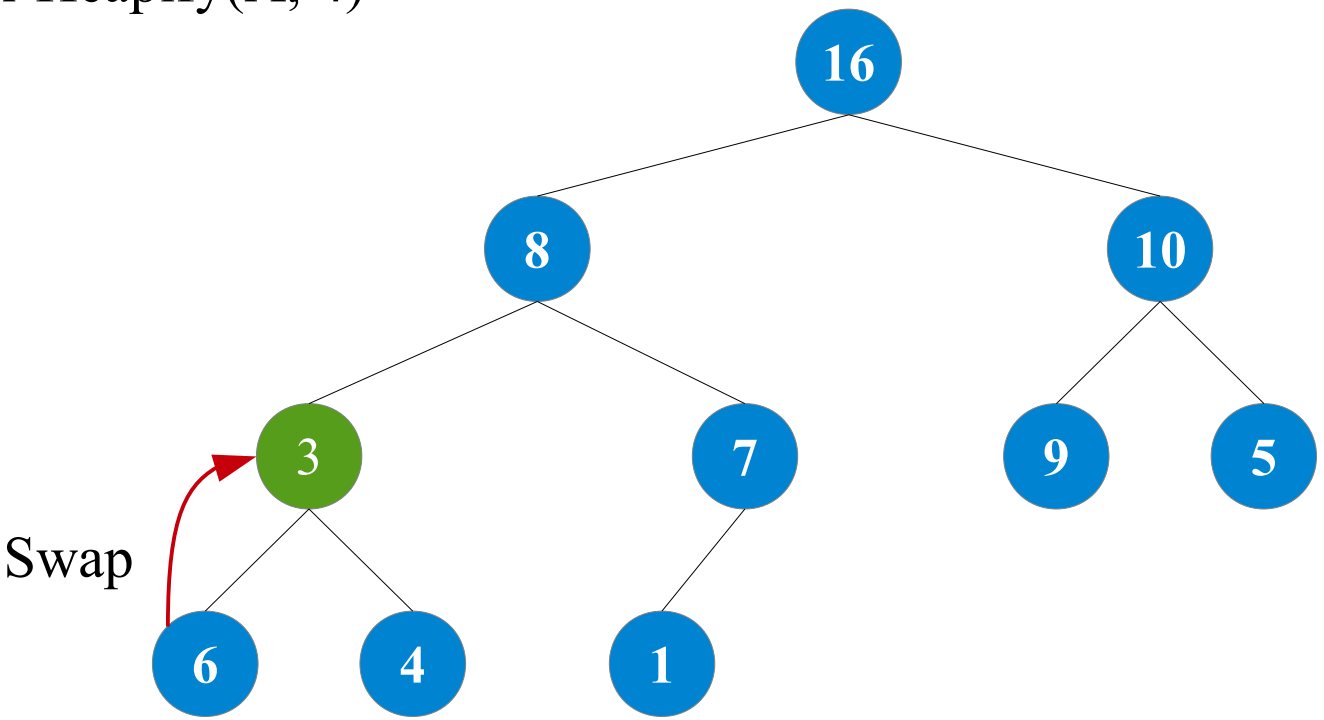
Max-Heapify(A, 4)



1	2	3	4	5	6	7	8	9	10
16	8	10	3	7	9	5	6	4	1

Max-Heapify

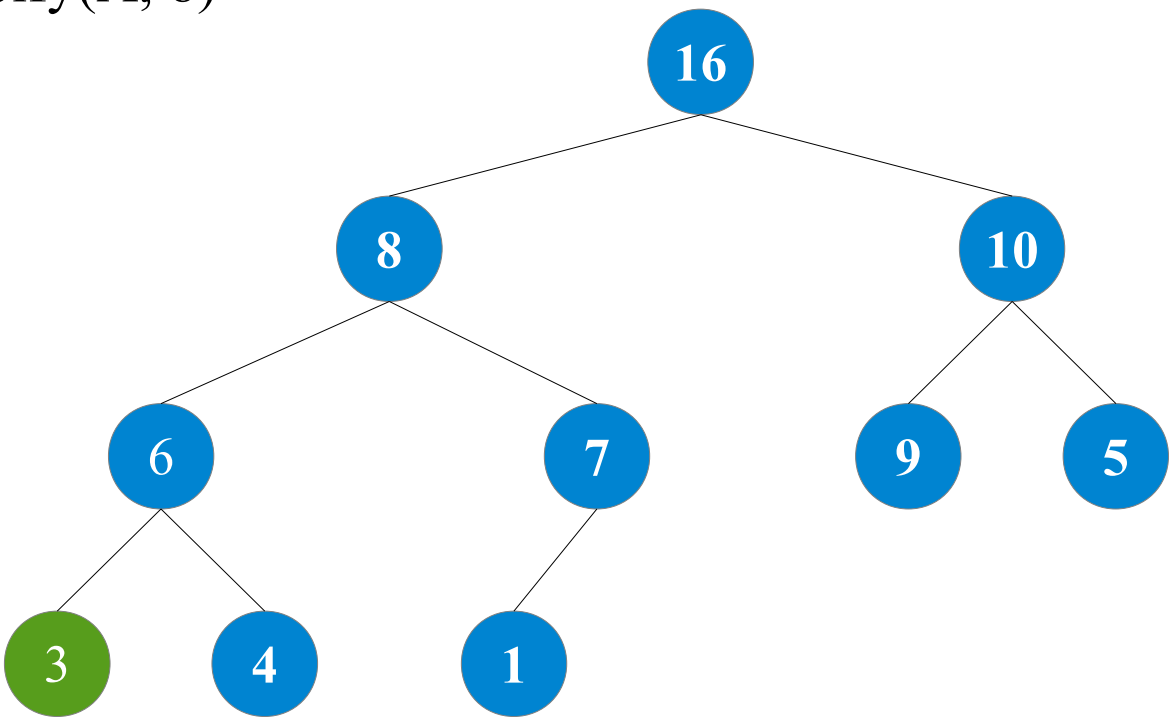
Max-Heapify(A, 4)



1	2	3	4	5	6	7	8	9	10
16	8	10	3	7	9	5	6	4	1

Max-Heapify

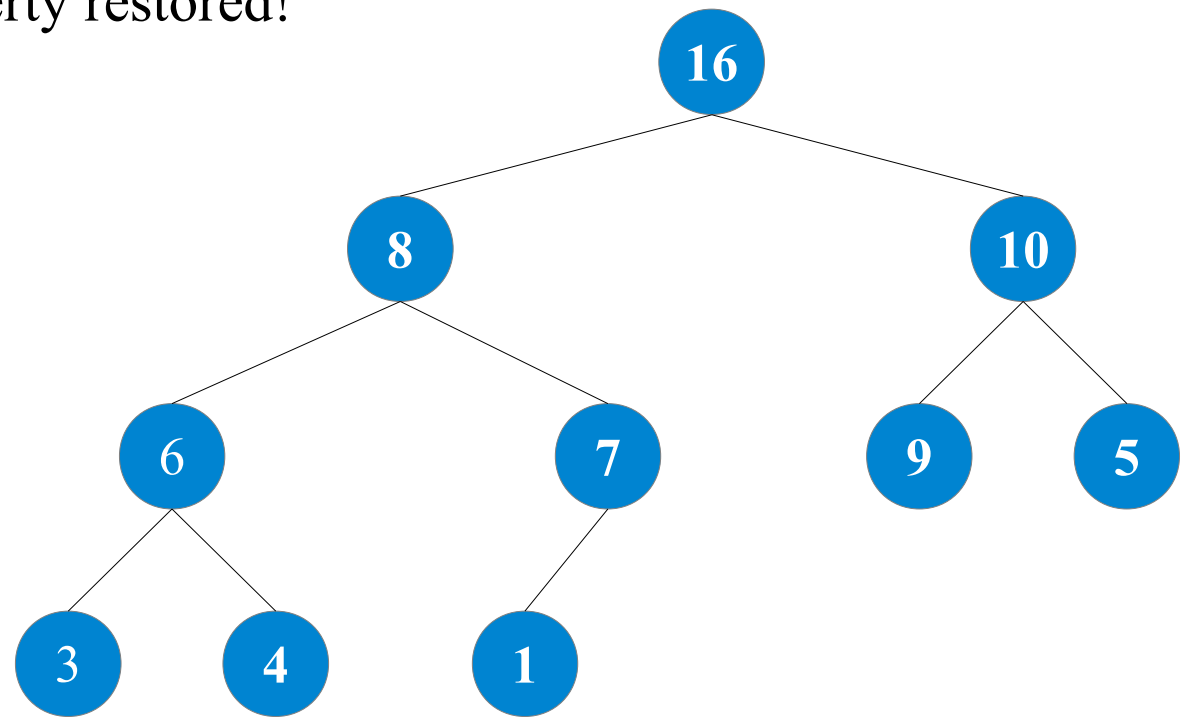
Max-Heapify(A, 8)



1	2	3	4	5	6	7	8	9	10
16	8	10	6	7	9	5	3	4	1

Max-Heapify

Heap property restored!



1	2	3	4	5	6	7	8	9	10
16	8	10	6	7	9	5	3	4	1

Max-Heapify Running Time

- Lines 1-9: $T(n) = \Theta(1)$
- Line 10: call on a subtree of node i. But what's the size of that subtree?

```
MAX-HEAPIFY (A, i)
1 l = LEFT(i)
2 r = RIGHT(i)
3 if l ≤ A.heap-size and A[l] > A[i]
4     largest = l
5 else largest = i
6 if r ≤ A.heap-size and A[r] > A[largest]
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```
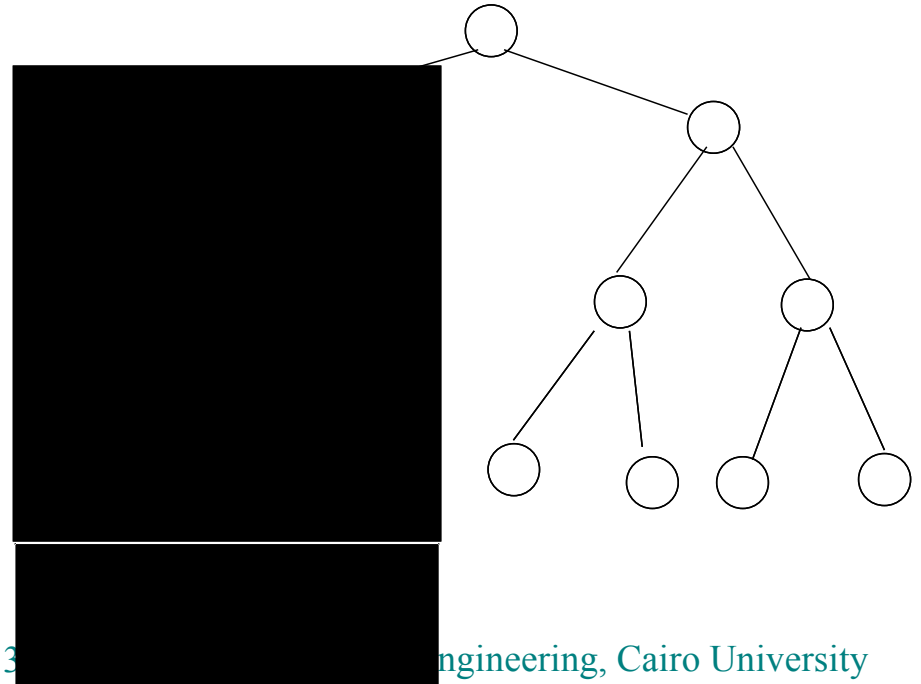

Max-Heapify Running Time

- Lines 1-9: $T(n) = \Theta(1)$
- Line 10: call on a subtree of node *i*. But what's the size of that subtree?

Best Case

In a complete binary tree, about half the nodes are included in the left subtree



15 nodes in tree

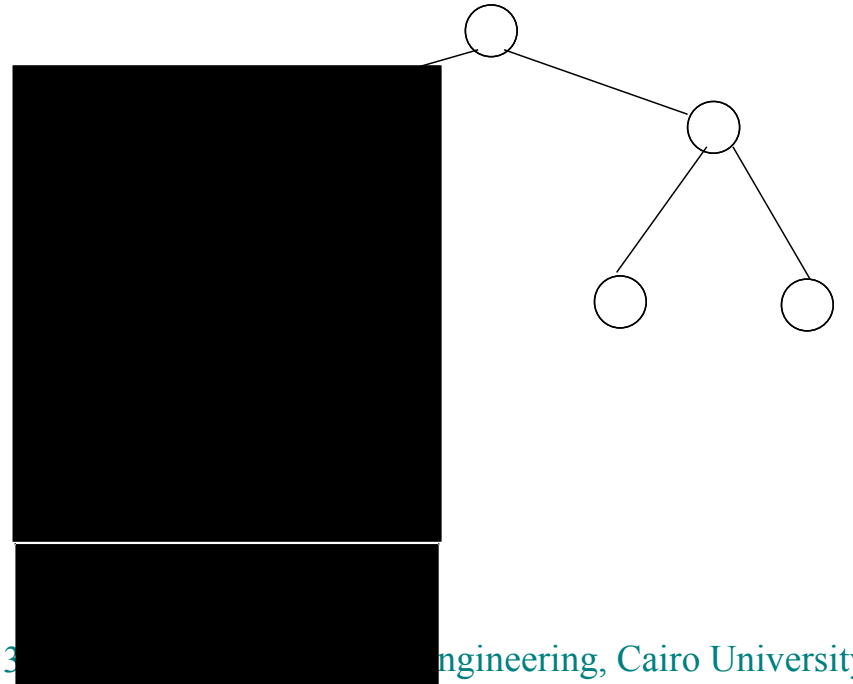
7 nodes in left subtree

Max-Heapify Running Time

- Lines 1-9: $T(n) = \Theta(1)$
- Line 10: call on a subtree of node *i*. But what's the size of that subtree?

Worst Case

In a binary tree where bottom is half-full, about two-thirds of the nodes are included in the left subtree



11 nodes in tree

7 nodes in left subtree

Max-Heapify Running Time

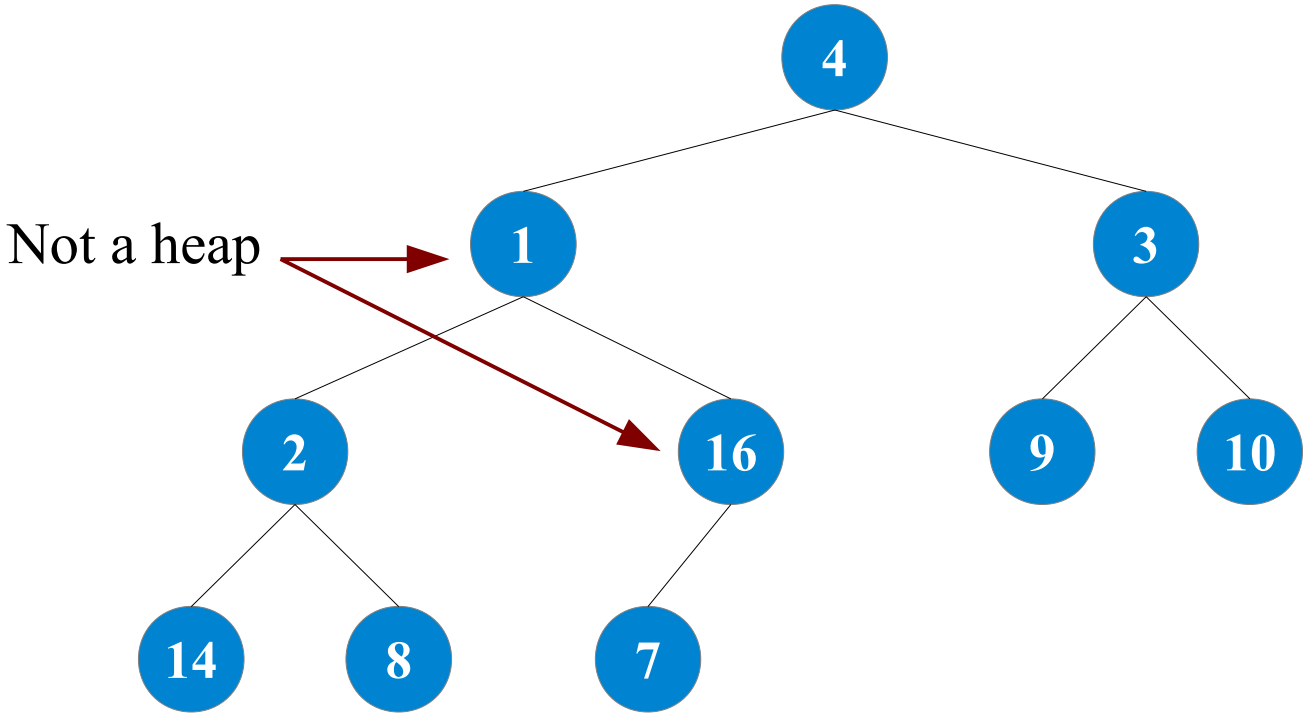
- $T(n) \leq T(2n/3) + \Theta(1)$
- Case 2 of the Master Theorem ($a = 1$, $b = 3/2$):
  - $T(n) = O(\lg n)$
- Can also represent it as a function of h, the height of the node
  - The height is the number of links from the node to the leaves
  - $T(n) = O(h)$

Heap Building

```
Build-MAX-HEAP(A)
1  A.heap-size = A.length
2  for i = ⌊A.length / 2⌋ downto 1
3     MAX-HEAPIFY(A, i)
```

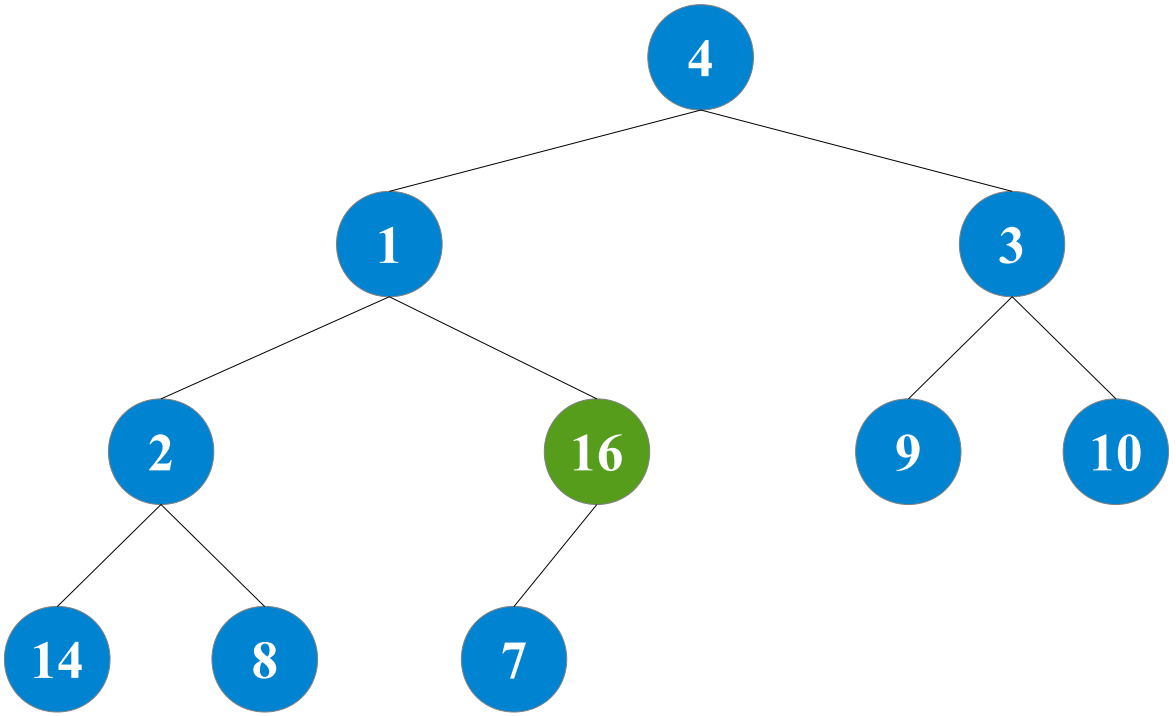
- Each leaf is already a heap, albeit a useless one!
- Start from the second to last level, and heapify i.e. move values violating the Heap property downwards
- Ends with a Max-Heap!

Heap Building



1	2	3	4	5	6	7	8	9	10
4	1	3	2	16	9	10	14	8	7

Heap Building



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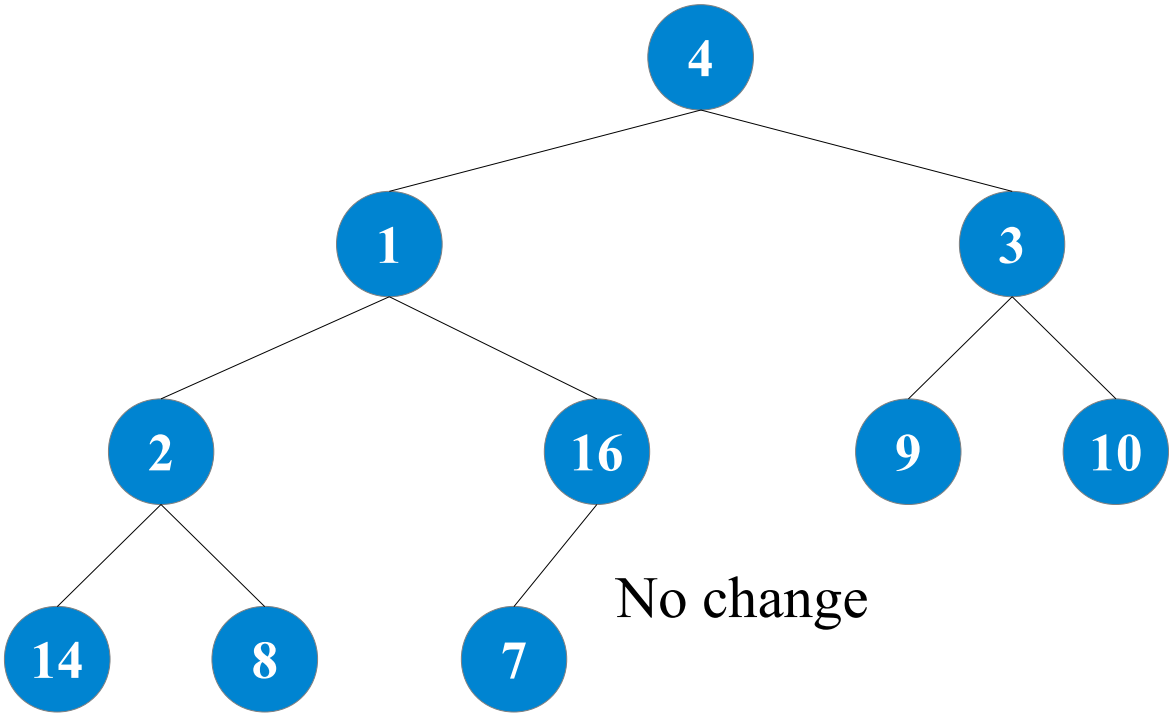
Heap Building



1	2	3	4	5	6	7	8	9	10
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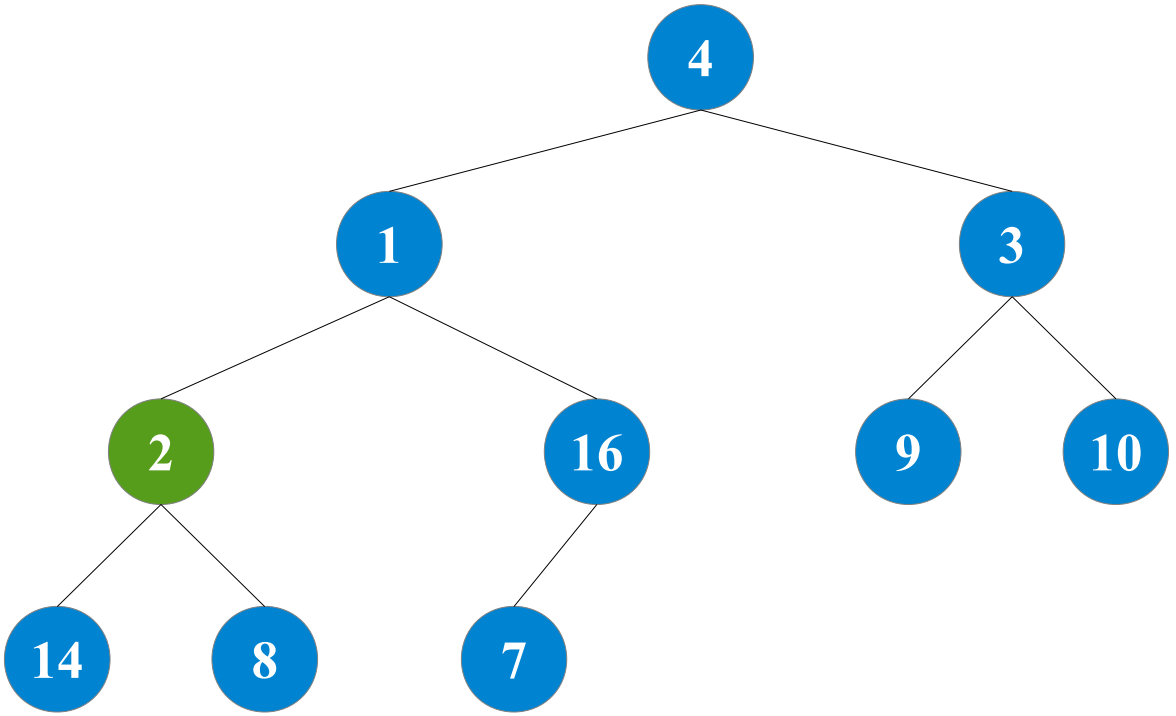
Heap Building



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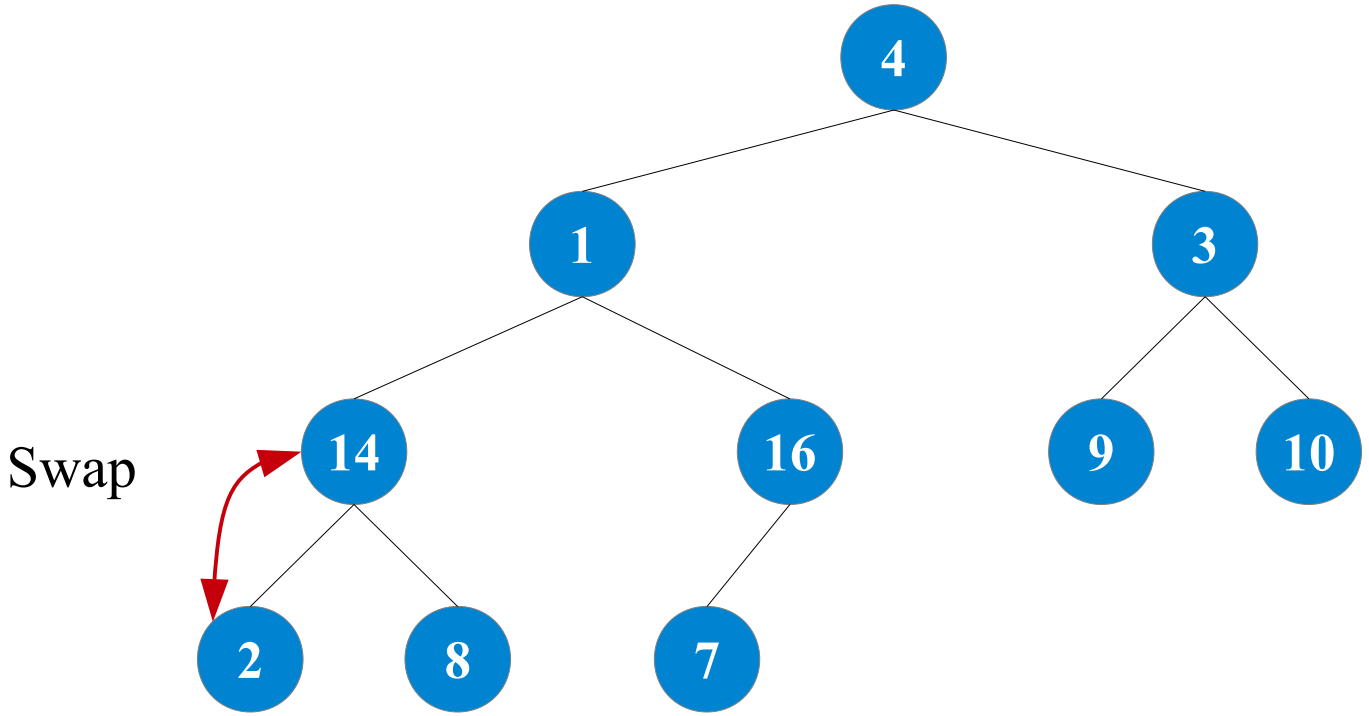
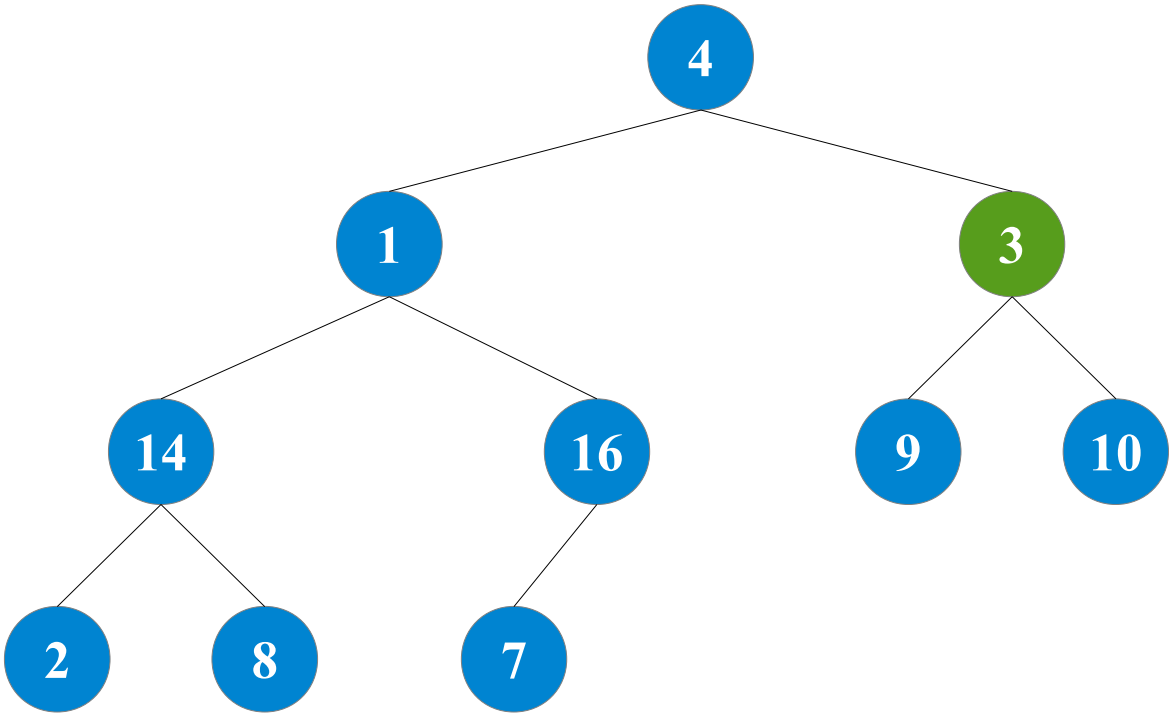
Heap Building



1	2	3	4	5	6	7	8	9	10
4	1	3	14	16	9	10	2	8	7

Heap Building



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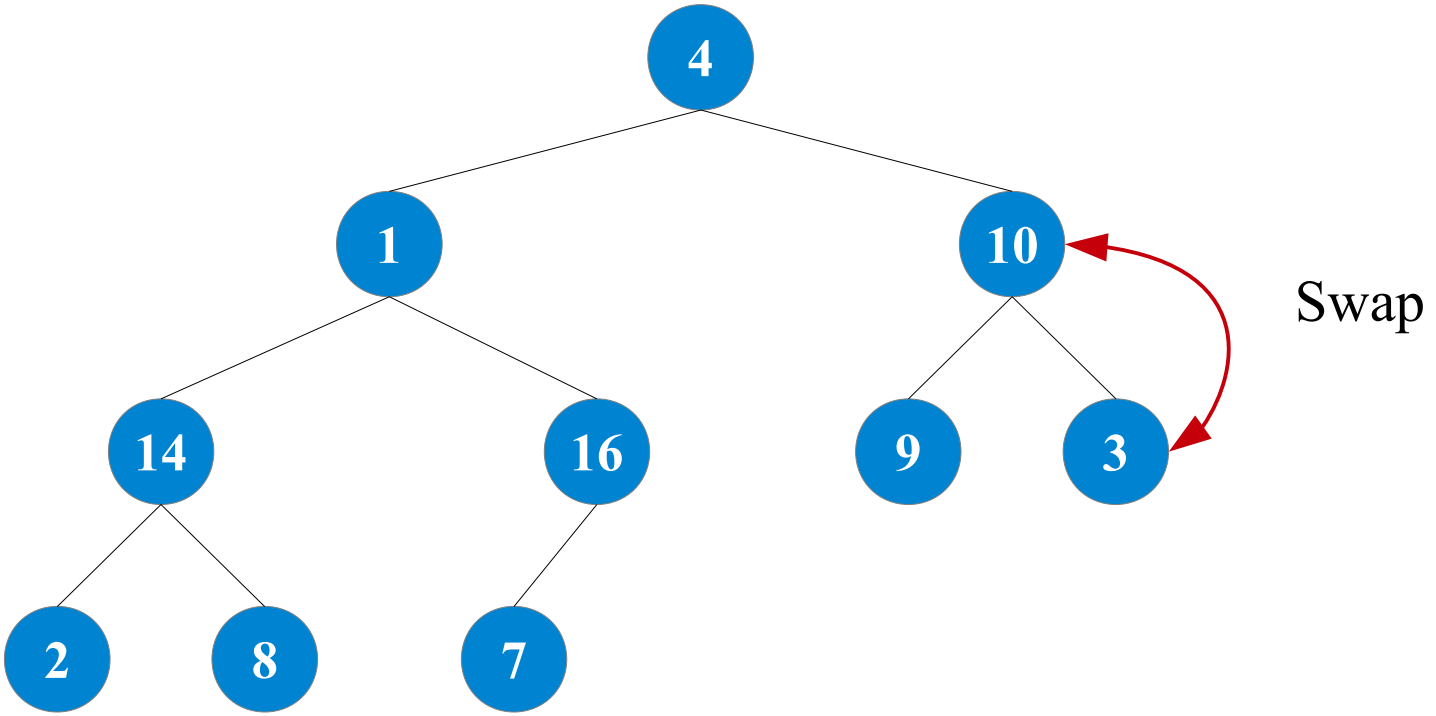
Heap Building



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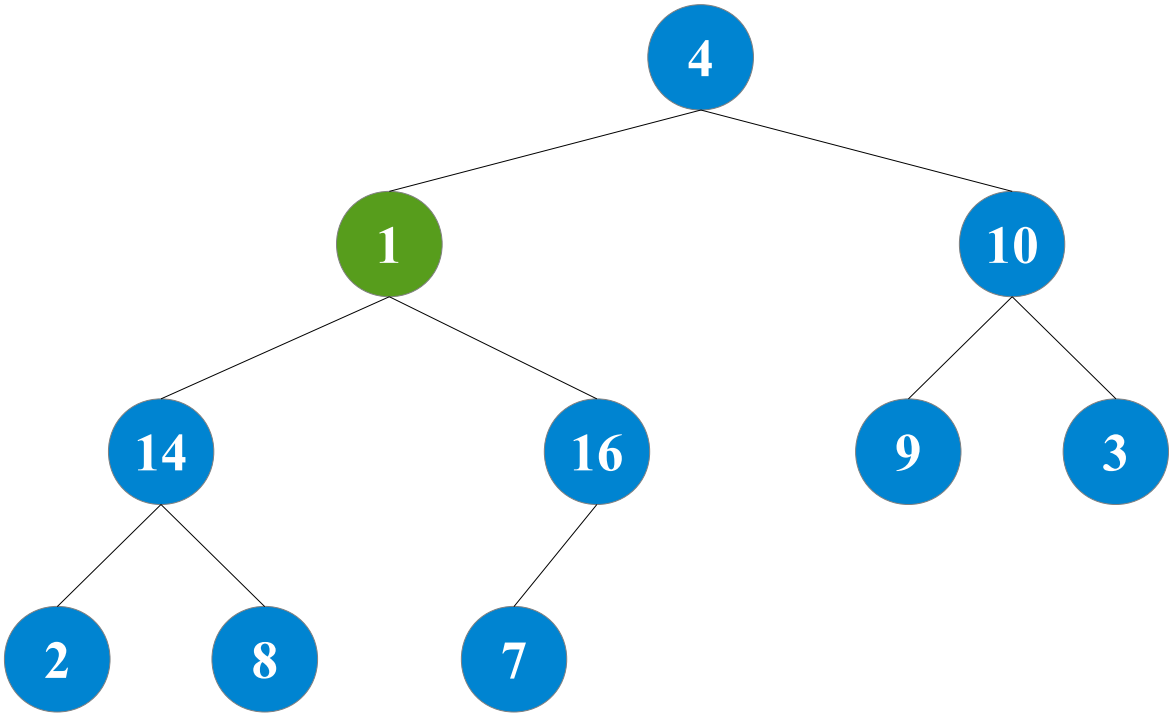
Heap Building



1	2	3	4	5	6	7	8	9	10
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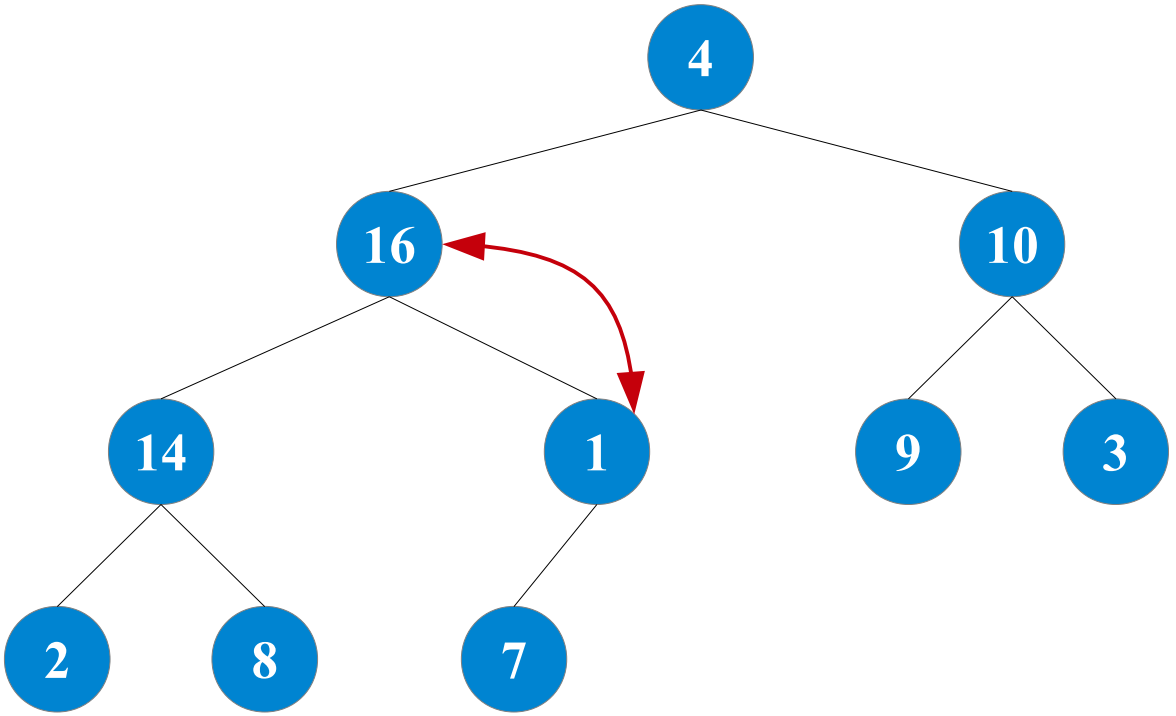
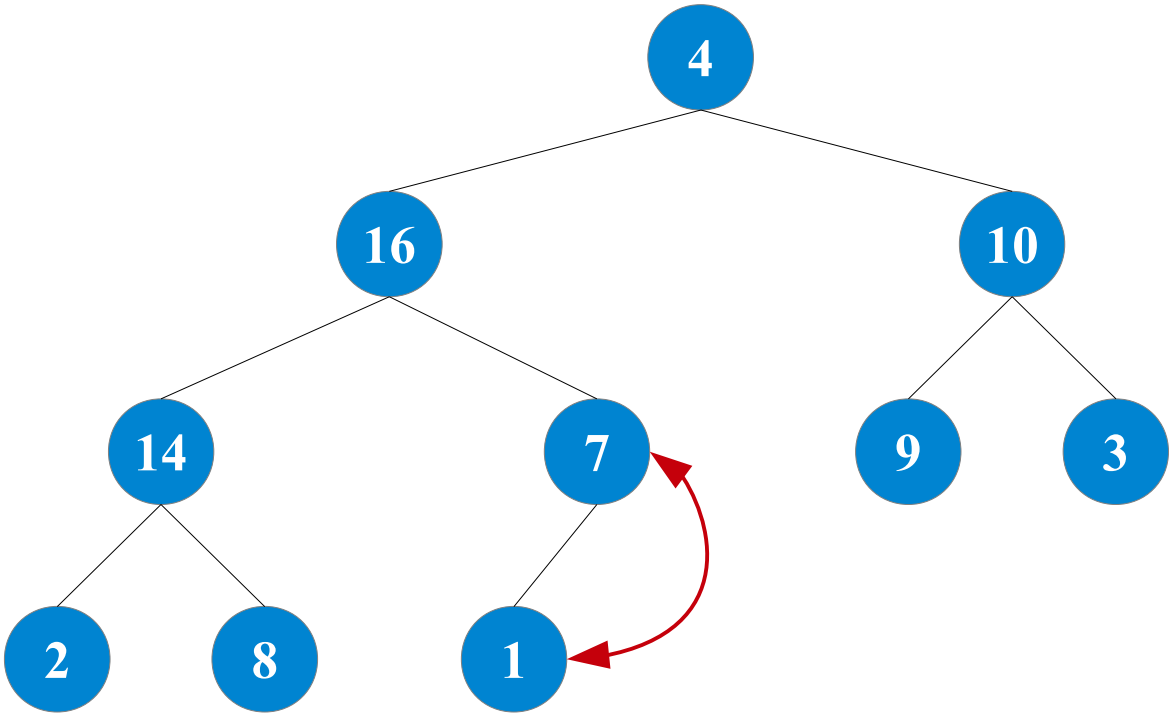
Heap Building



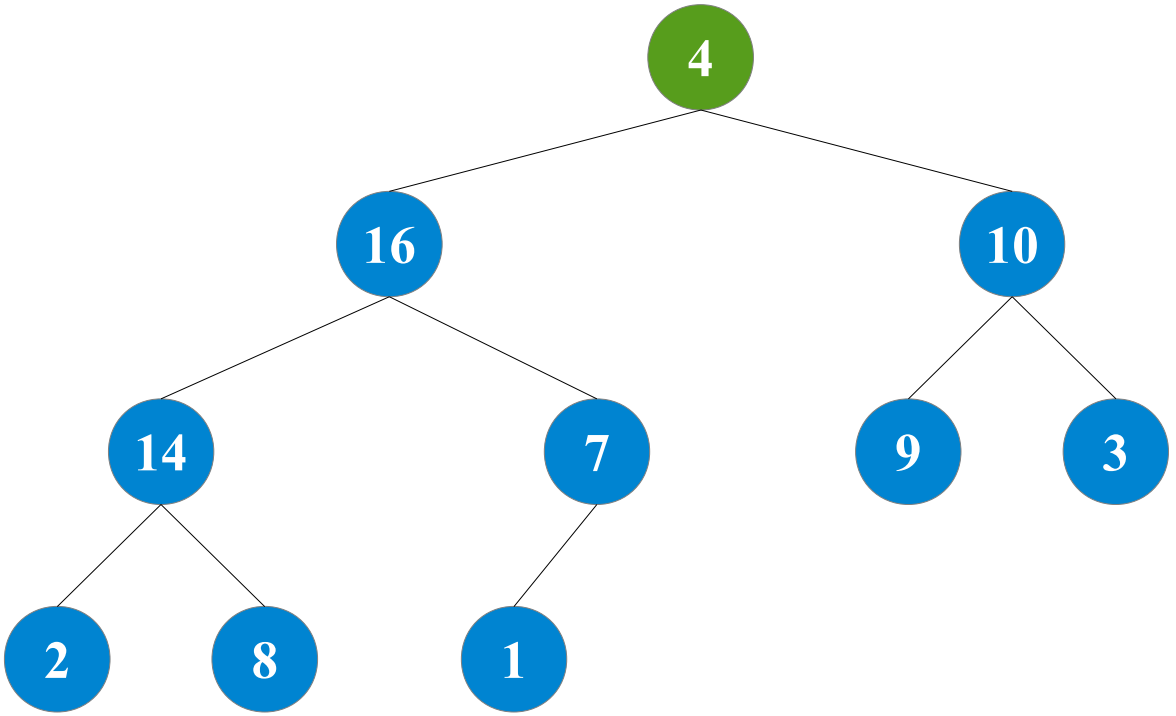
1	2	3	4	5	6	7	8	9	10
4	16	10	14	1	9	3	2	8	7

Heap Building



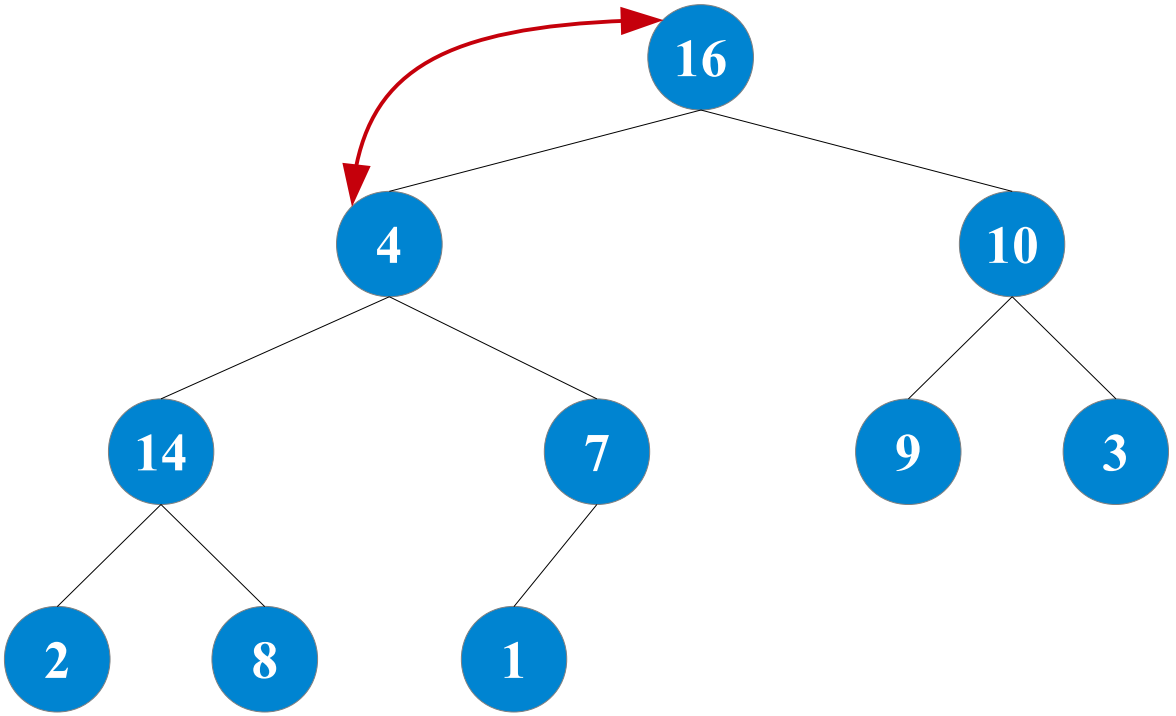
1	2	3	4	5	6	7	8	9	10
4	16	10	14	7	9	3	2	8	1

Heap Building



1	2	3	4	5	6	7	8	9	10
4	16	10	14	7	9	3	2	8	1

Heap Building



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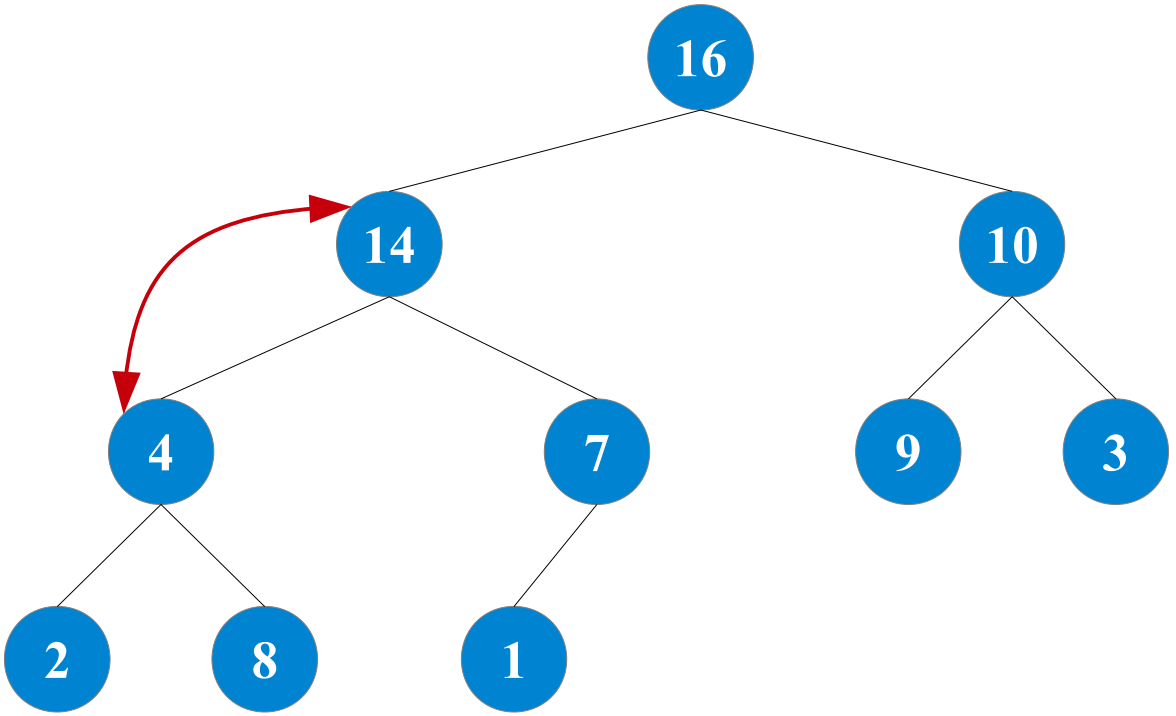
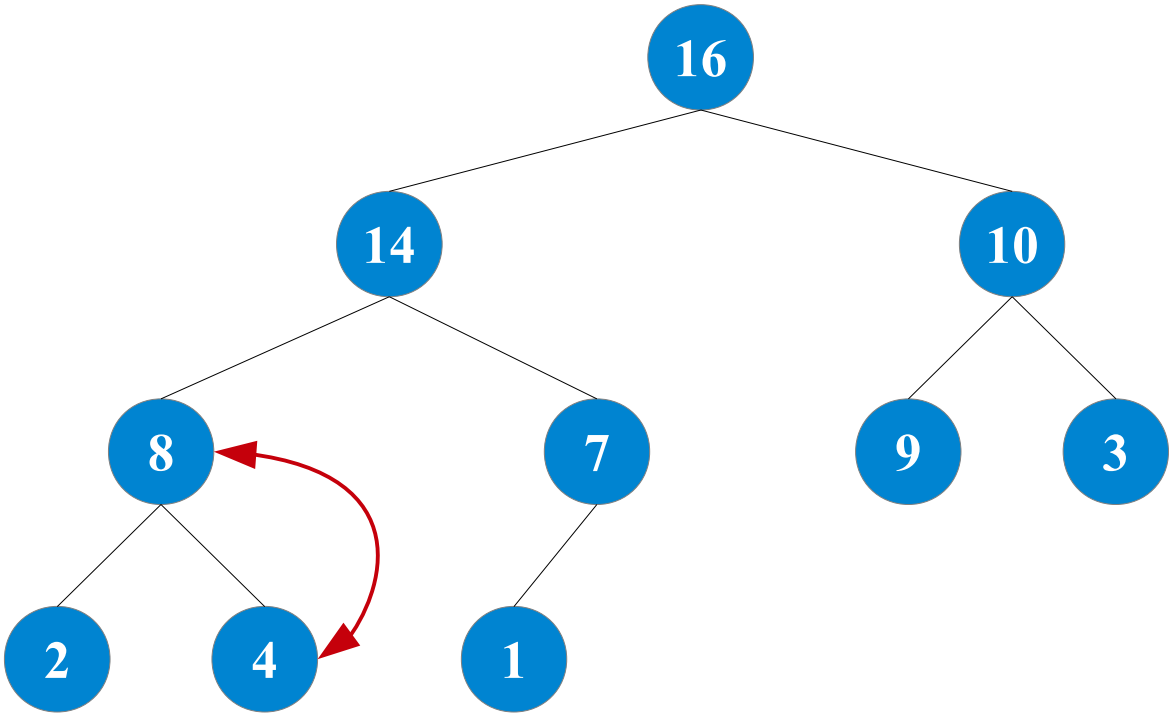
16	4	10	14	7	9	3	2	8	1

Heap Building



1	2	3	4	5	6	7	8	9	10
16	14	10	4	7	9	3	2	8	1

Heap Building



1	2	3	4	5	6	7	8	9	10
16	14	10	8	7	9	3	2	4	1

Build-Max-Heap Running Time

- Loose upper bound:
  - $n/2$ calls to Max-Heapify, each with time $\lg n$
  - $T(n) = O(n \lg n)$
- Tighter upper bound:
  - $T(n) = O(n)$

$$T(n) = \sum_{h=0}^{floor(\lg n)} O(h)\, ceil\left(\frac{n}{2^{h+1}}\right)$$

$$T(n) = O\left(n \sum_{h=0}^{floor(\lg n)} \frac{h}{2^h}\right)$$

$\sum_{h=0}^{\infty} k\, x^k = \frac{x}{1-x}$ Why? $\frac{d}{dx}\left(\sum_{h=0}^{\infty} x^k = \frac{1}{1-x}\right)$ → $\sum_{h=0}^{\infty} \frac{h}{2^h} = \frac{1/2}{1-1/2} = 1$

$$T(n) = O(n)$$

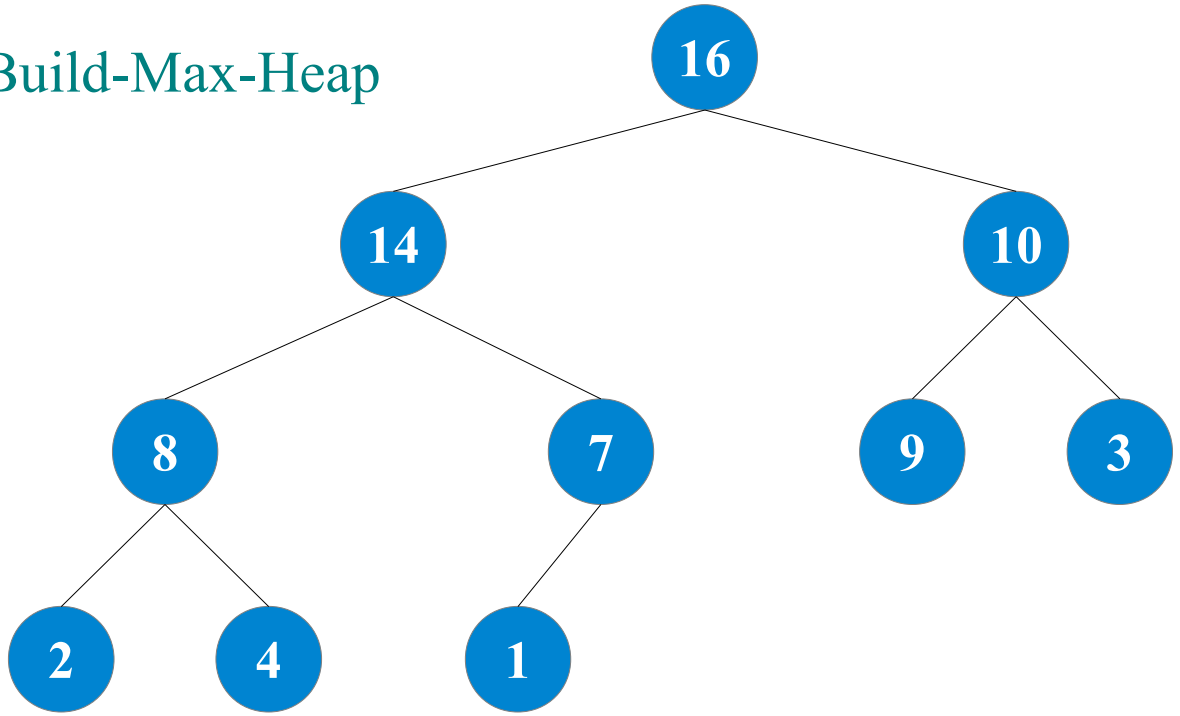
Heapsort

```
HEAPSORT(A)
1  BUILD-MAX-HEAP(A)
2  for i = A.length downto 1
3     exchange A[i] with A[1]
4     A.heap-size --
5     MAX-HEAPIFY(A, i)
```

- Build a Max-Heap from the array → put the maximum value at the front of A
- Put the max value in its right sorted place at the end and the value at the end in the root
- Heapify and repeat ...

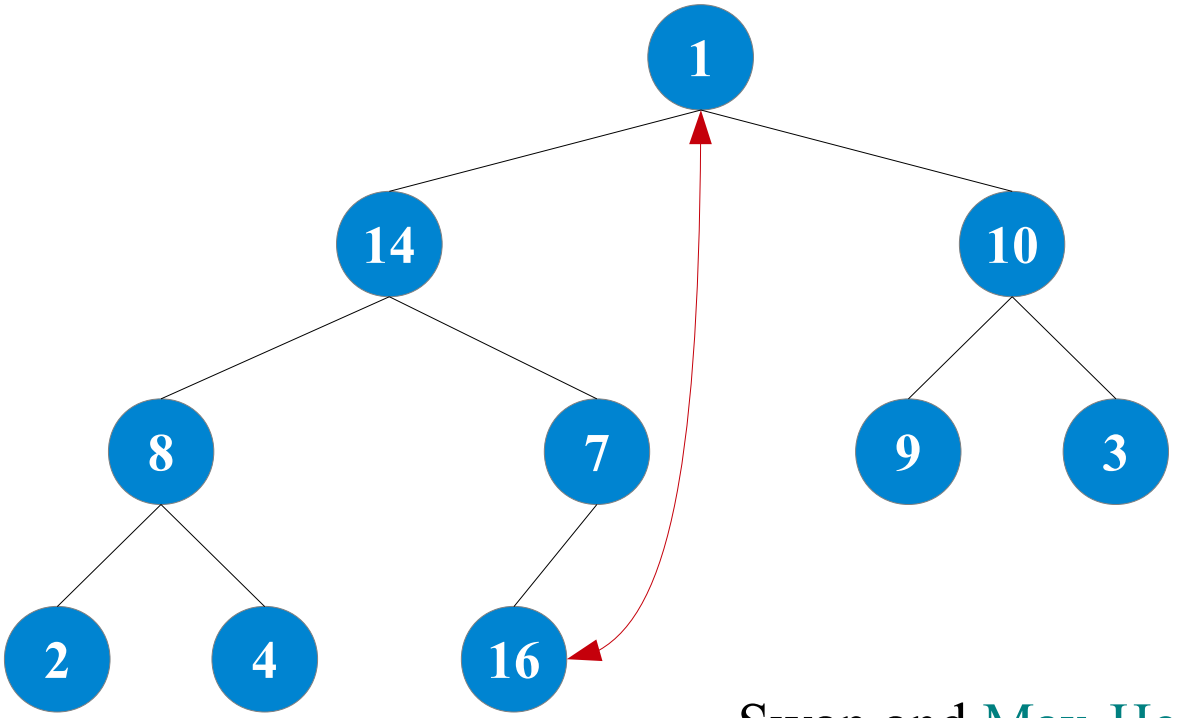
Heapsort

After calling Build-Max-Heap



1	2	3	4	5	6	7	8	9	10
16	14	10	8	7	9	3	2	4	1

Heapsort

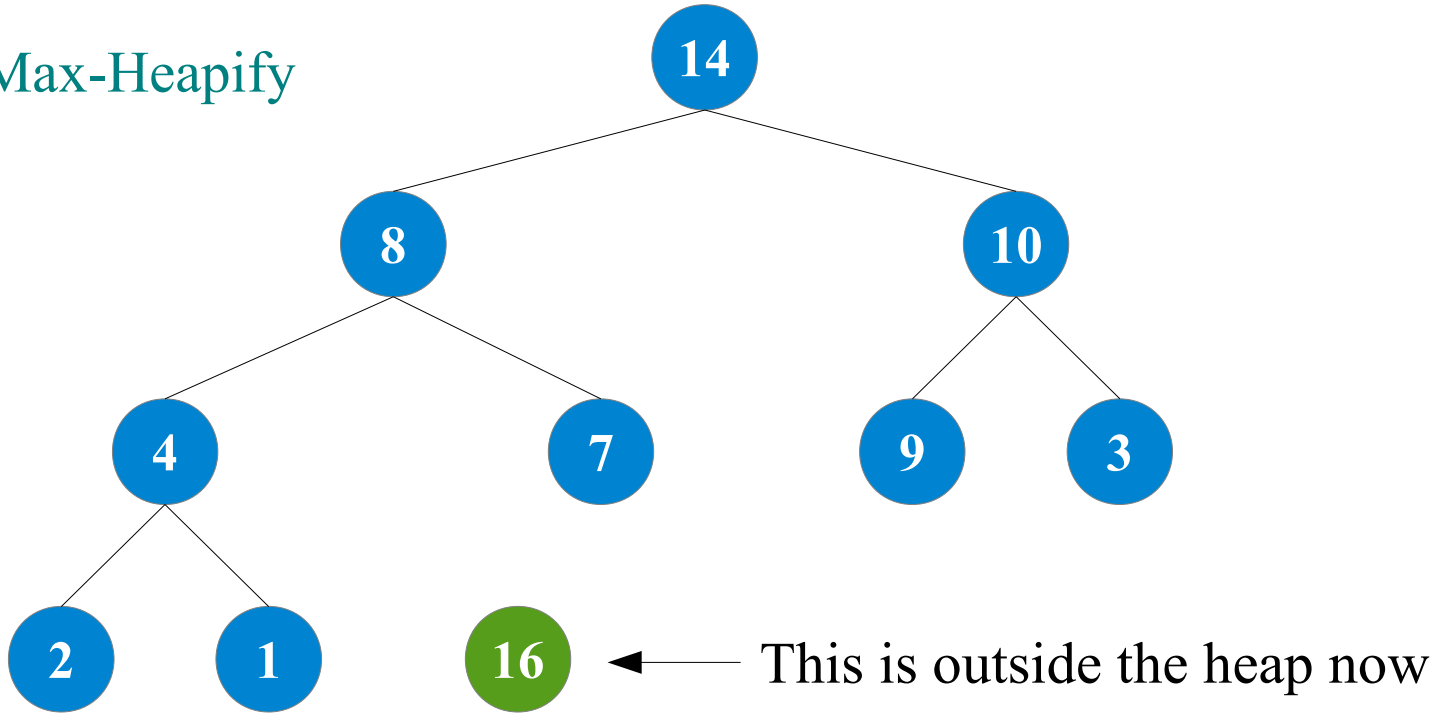


Swap and Max-Heapify

1	2	3	4	5	6	7	8	9	10
1	14	10	8	7	9	3	2	4	16

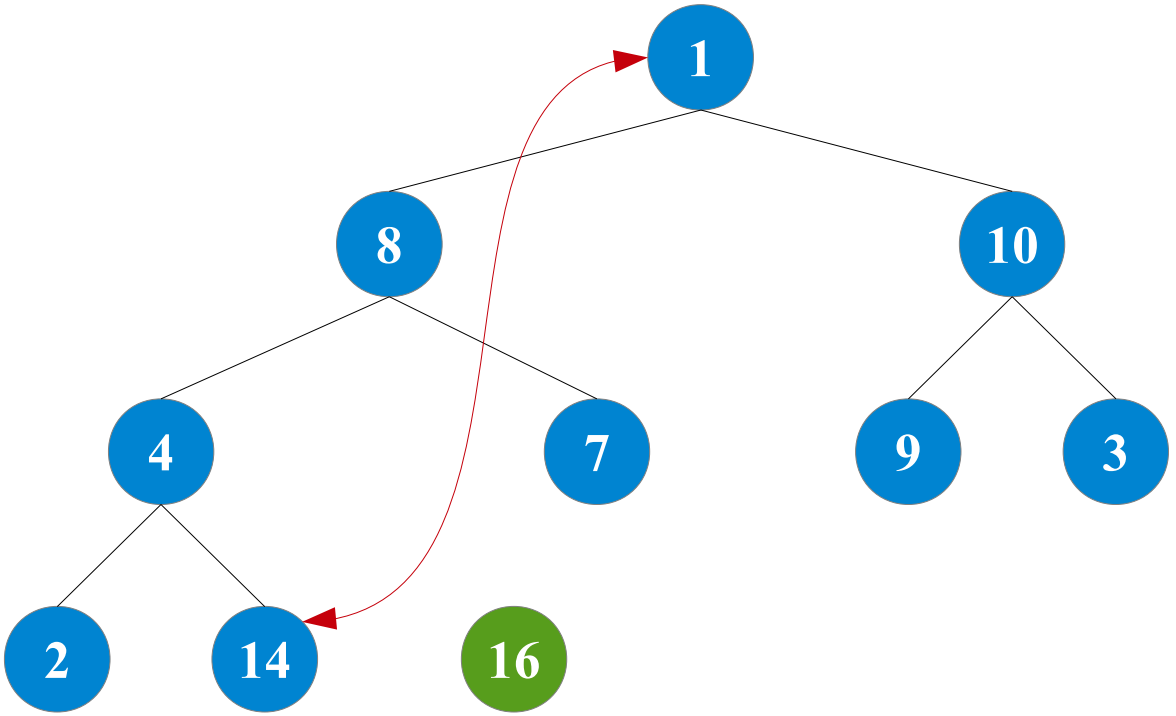
Heapsort

After calling Max-Heapify



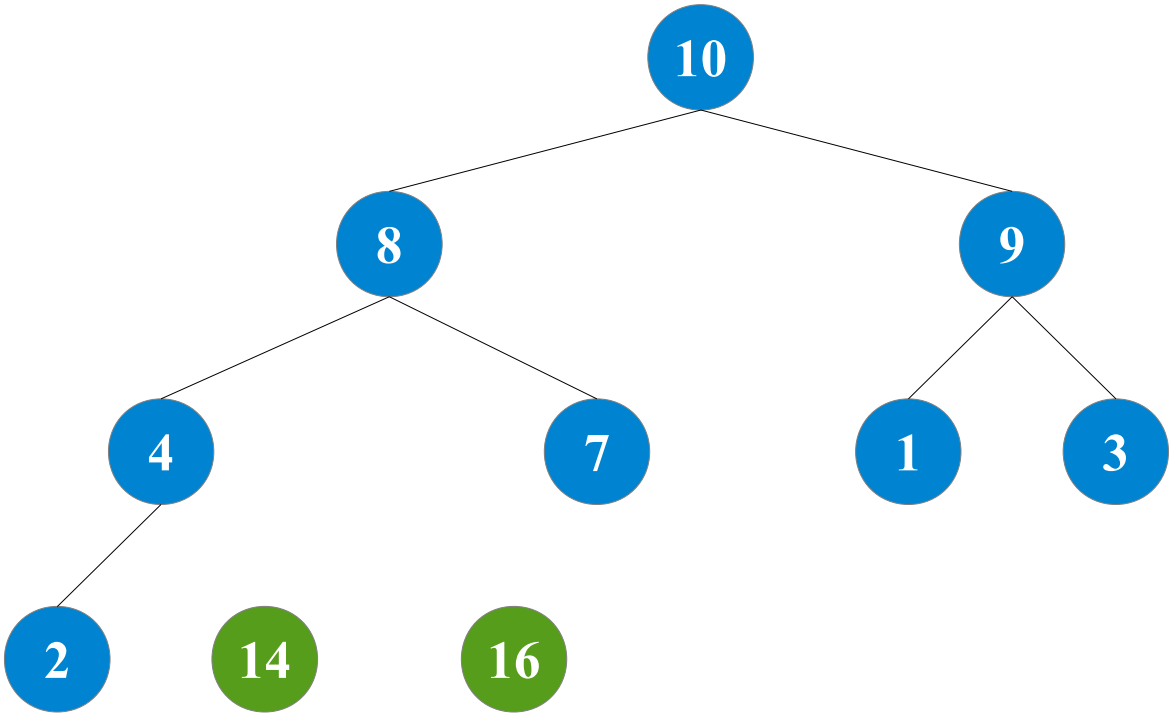
1	2	3	4	5	6	7	8	9	10
14	8	10	4	7	9	3	2	1	16

Heapsort



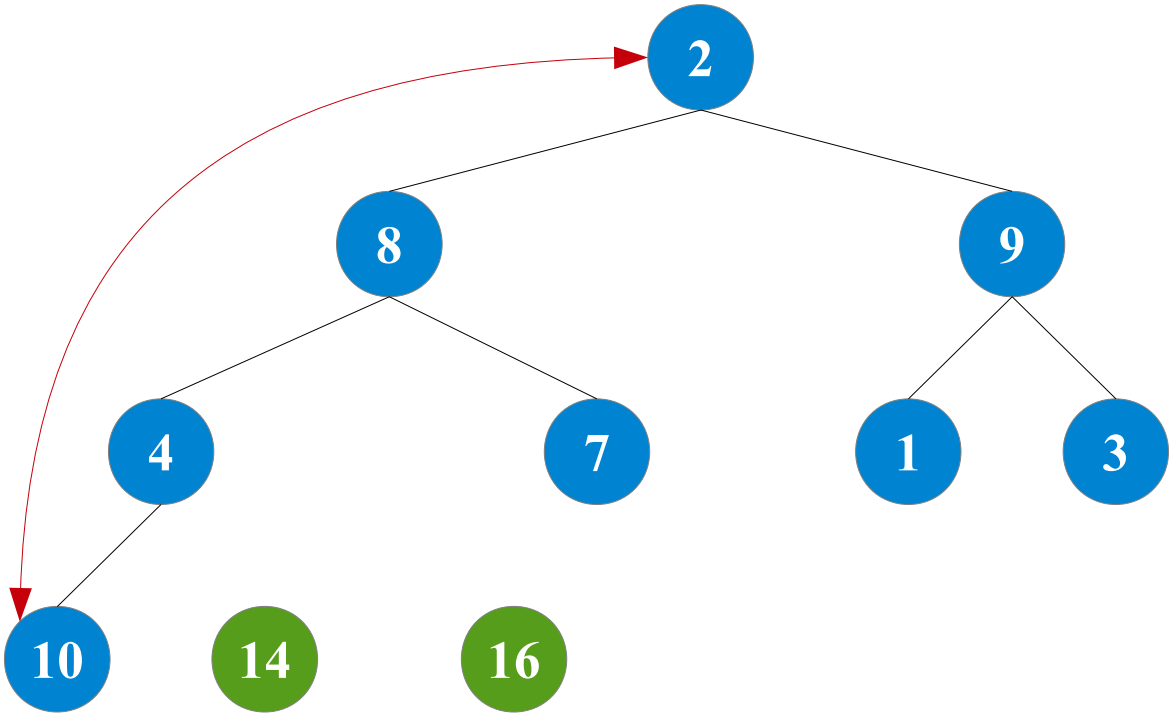
1	2	3	4	5	6	7	8	9	10
1	8	10	4	7	9	3	2	14	16

Heapsort



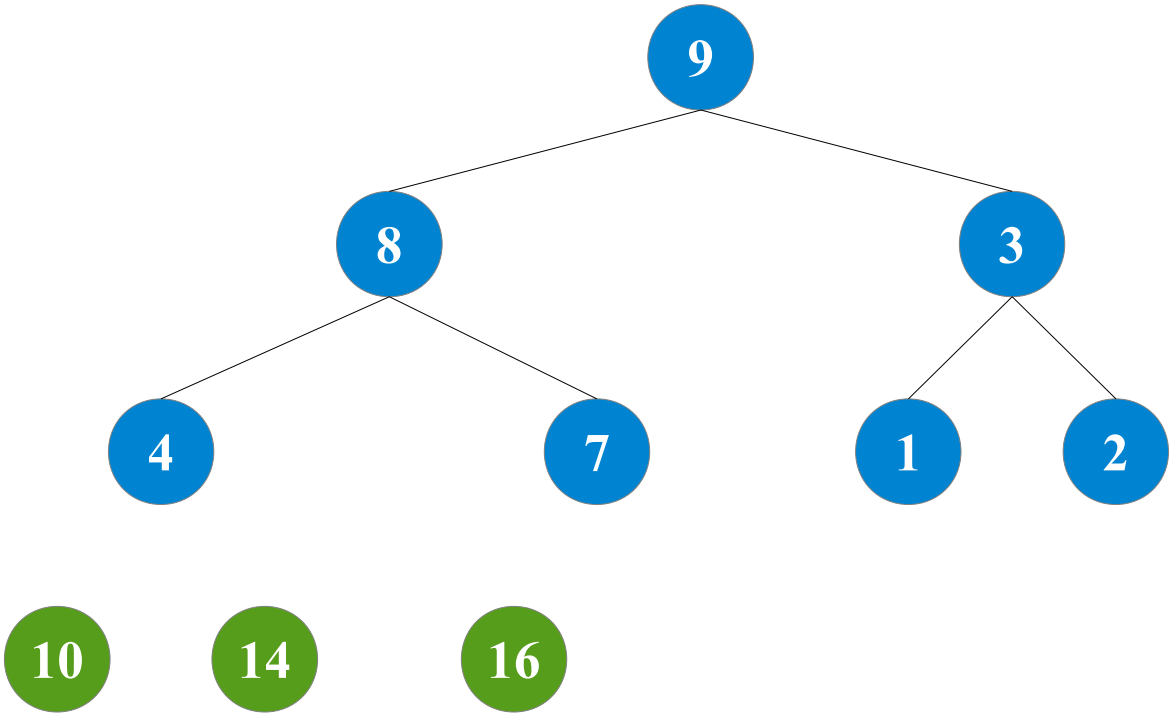
1	2	3	4	5	6	7	8	9	10
10	8	9	4	7	1	3	2	14	16

Heapsort



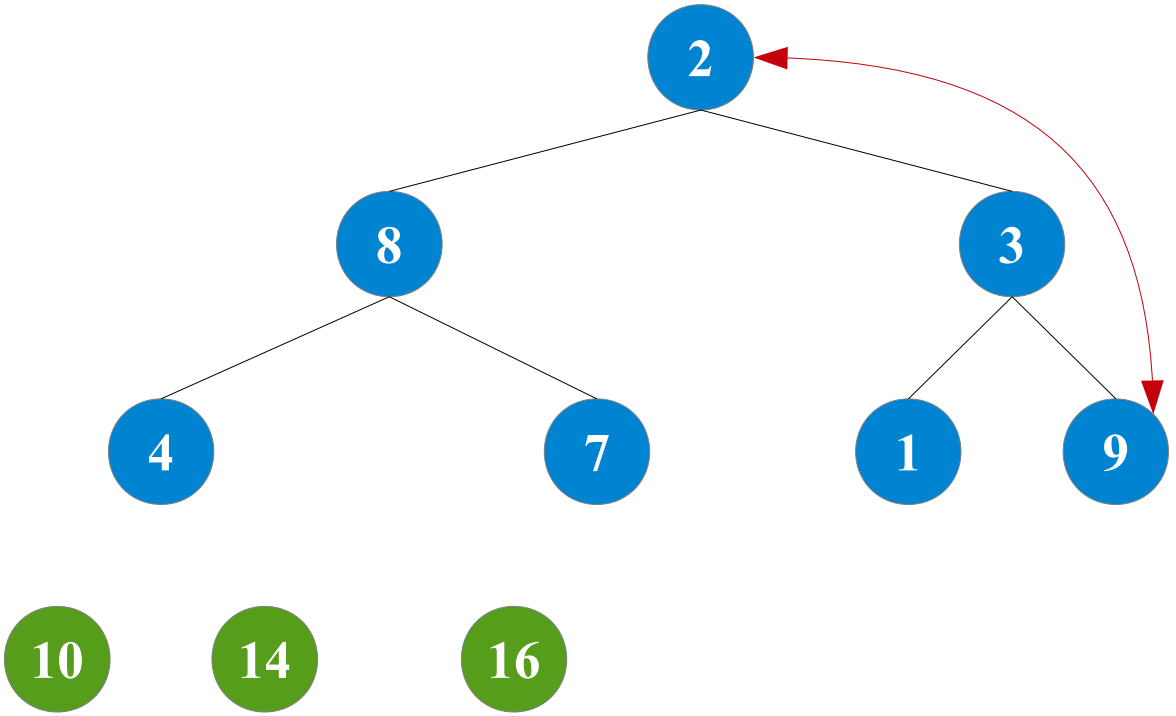
1	2	3	4	5	6	7	8	9	10
2	8	9	4	7	1	3	10	14	16

Heapsort



1	2	3	4	5	6	7	8	9	10
9	8	3	4	7	1	2	10	14	16

Heapsort



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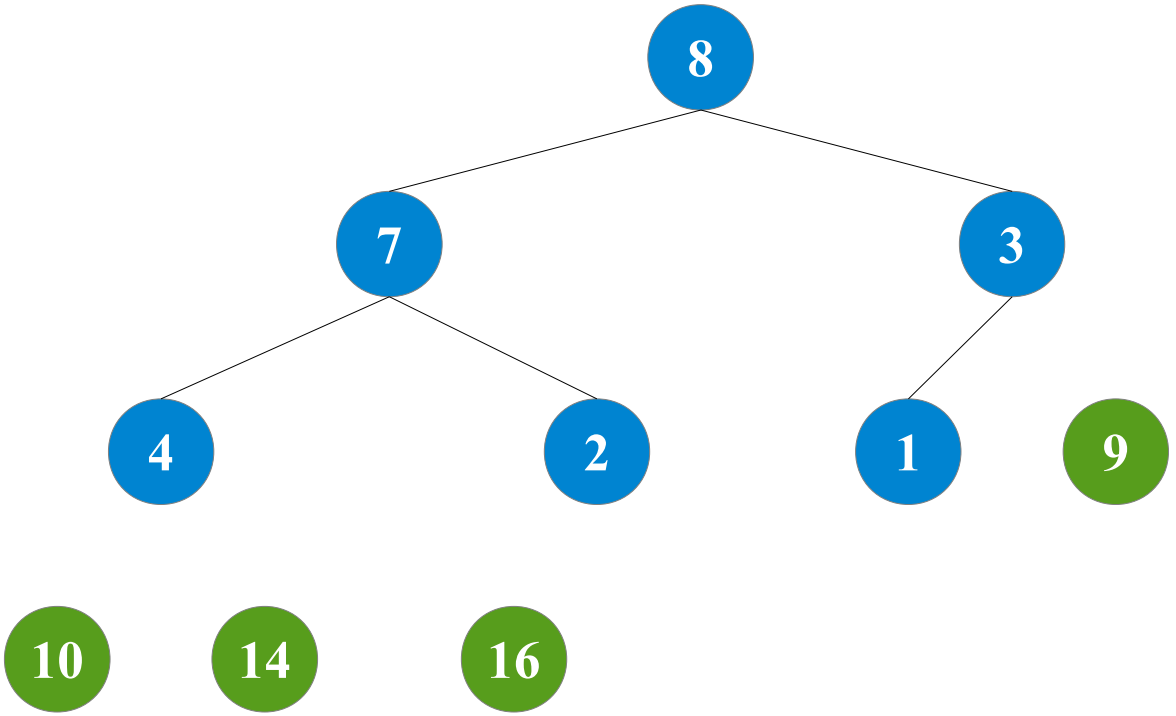

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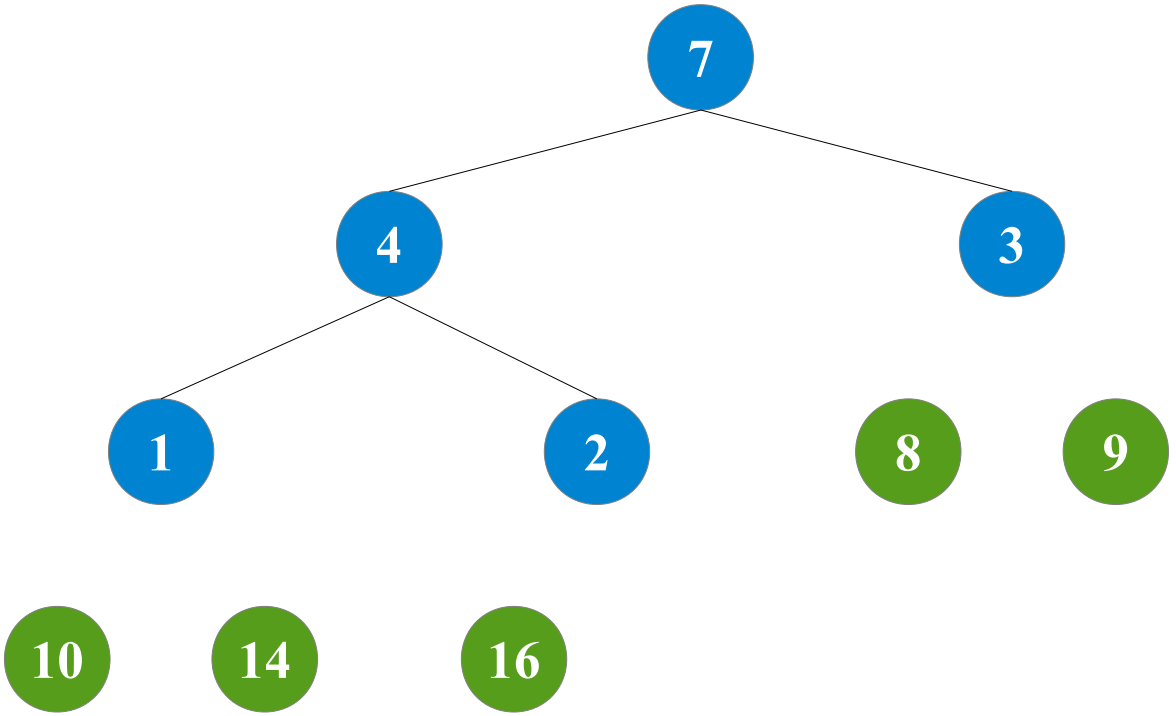
Heapsort



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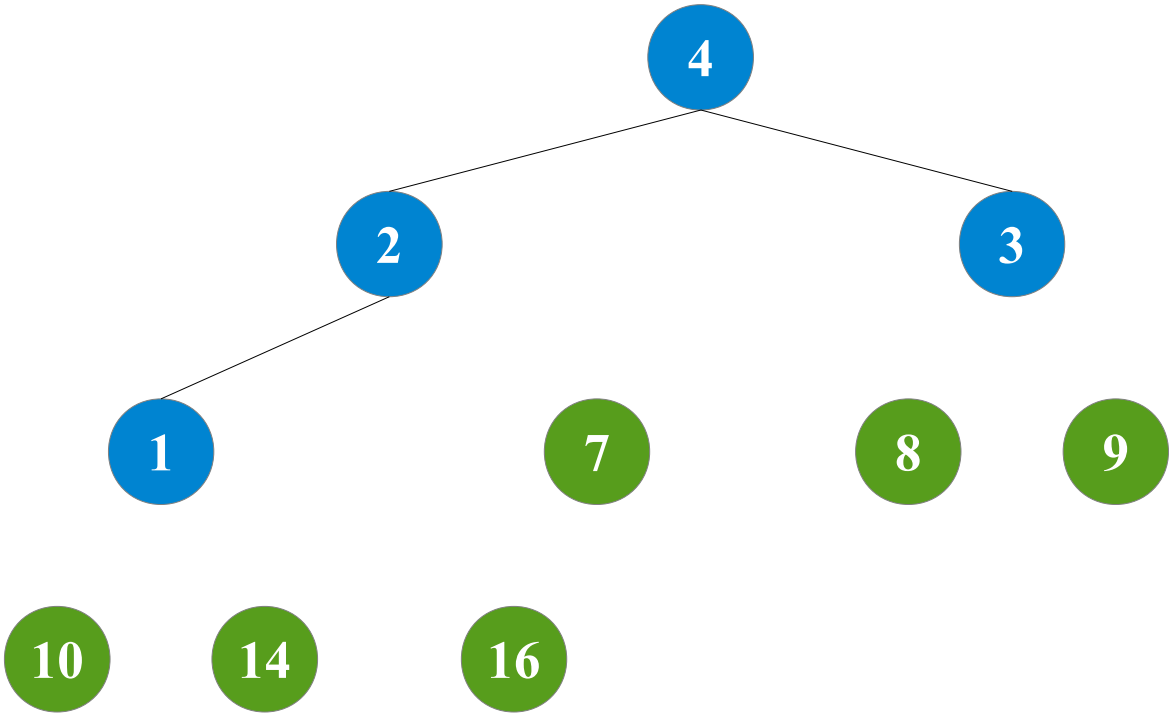
Heapsort



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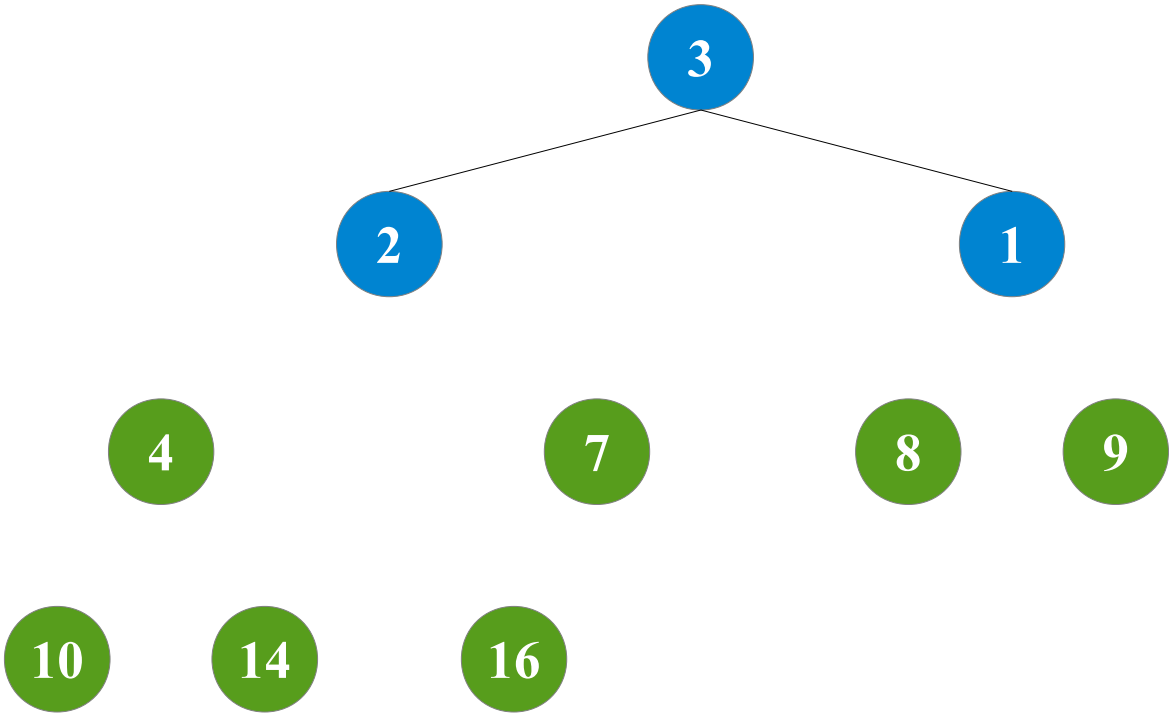
Heapsort



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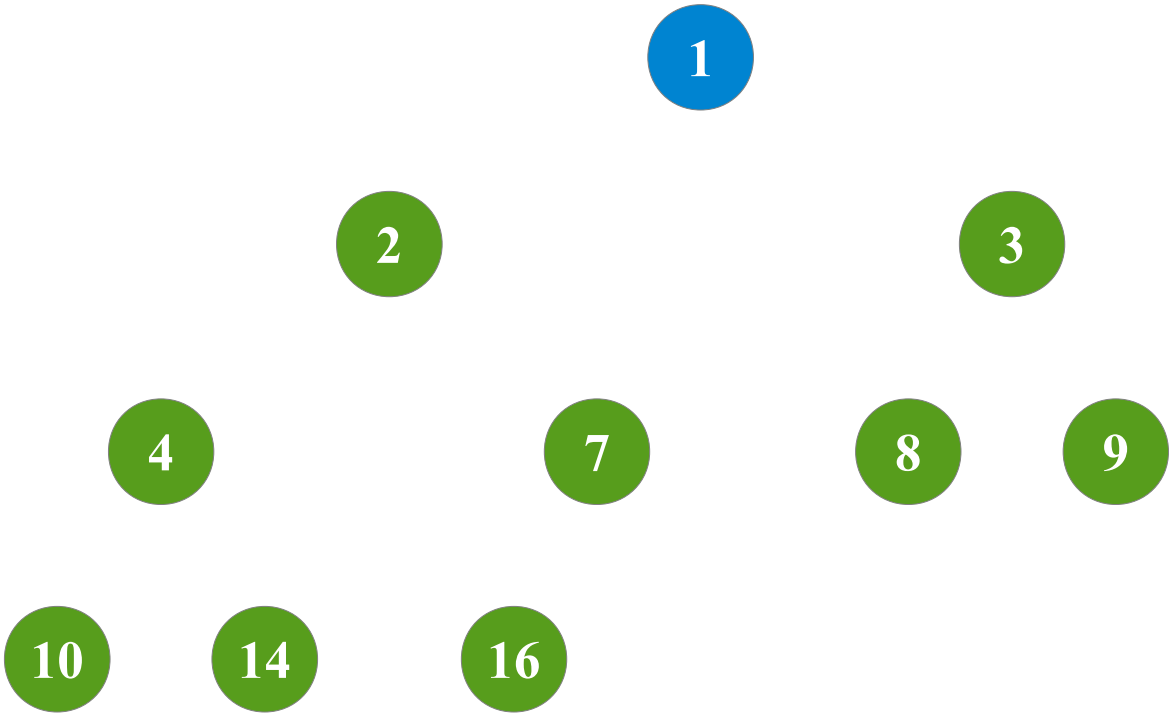
Heapsort



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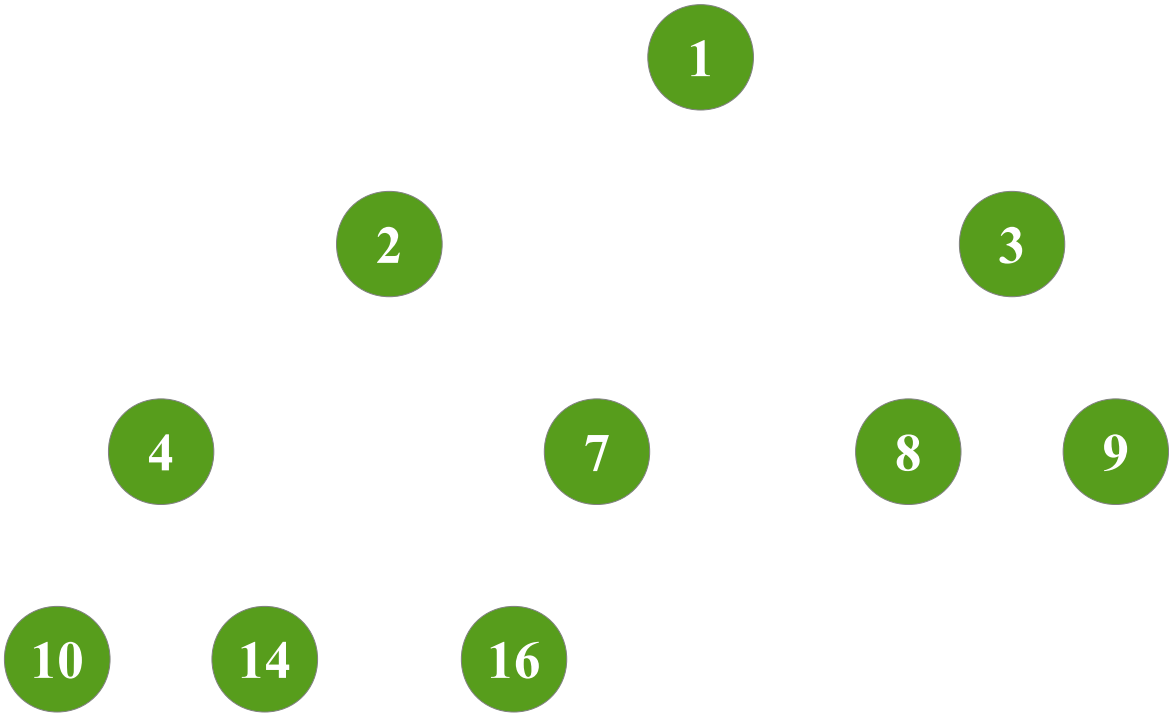
Heapsort



1	2	3	4	5	6	7	8	9	10
1	2	3	4	7	8	9	10	14	16

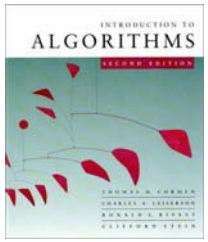
Heapsort Running time

- T(n) = $O(n)$ for Build-Max-Heap +
  n times $O(\lg n)$ for Max-Heapify
  = $O(n \lg n)$
- Constant space → "in place" sorting $\Theta(1)$
- What was Mergesort and Insertion sort?

```
HEAPSORT(A)
1  BUILD-MAX-HEAP(A)
2  for i = A.length downto 1
3     exchange A[i] with A[1]
4     A.heap-size --
5     MAX-HEAPIFY(A, i)
```

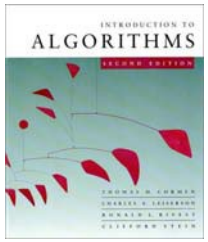
Priority Queues

- They are like FIFO (First-in First-out) Queues BUT depend on some priority value
- Elements with highest priority goes out first, even if the last one inserted!
- Applications include:
  - Event-driven simulation
  - Job scheduler etc.
- How is it implemented?
- Read the book



Quicksort

- Proposed by C.A.R. Hoare in 1962.
- Divide-and-conquer algorithm.
- Sorts "in place" (like insertion sort, but not like merge sort).
- Very practical (with tuning).



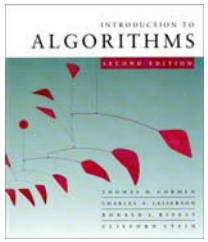
Divide and conquer

Quicksort an n-element array:

1. ***Divide:*** Partition the array into two subarrays around a ***pivot*** x such that elements in lower subarray $\leq x \leq$ elements in upper subarray.

2. ***Conquer:*** Recursively sort the two subarrays.
3. ***Combine:*** Trivial.

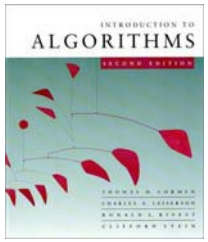
Key: *Linear-time partitioning subroutine.*



Pseudocode for quicksort

QUICKSORT(A, p, r)
  if $p < r$
    then $q \leftarrow$ PARTITION(A, p, r)
      QUICKSORT$(A, p, q-1)$
      QUICKSORT$(A, q+1, r)$

Initial call: QUICKSORT$(A, 1, n)$

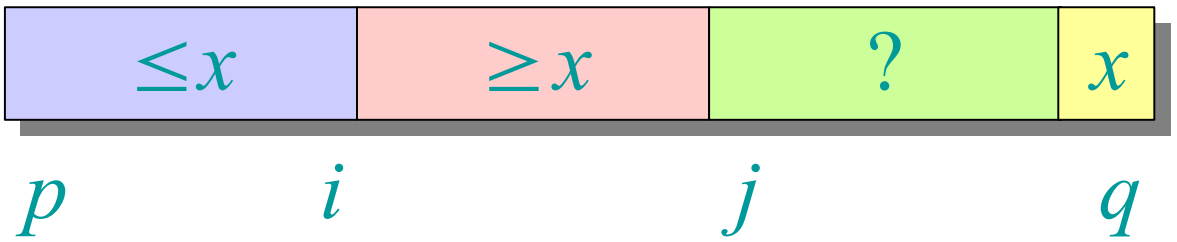


Partitioning subroutine

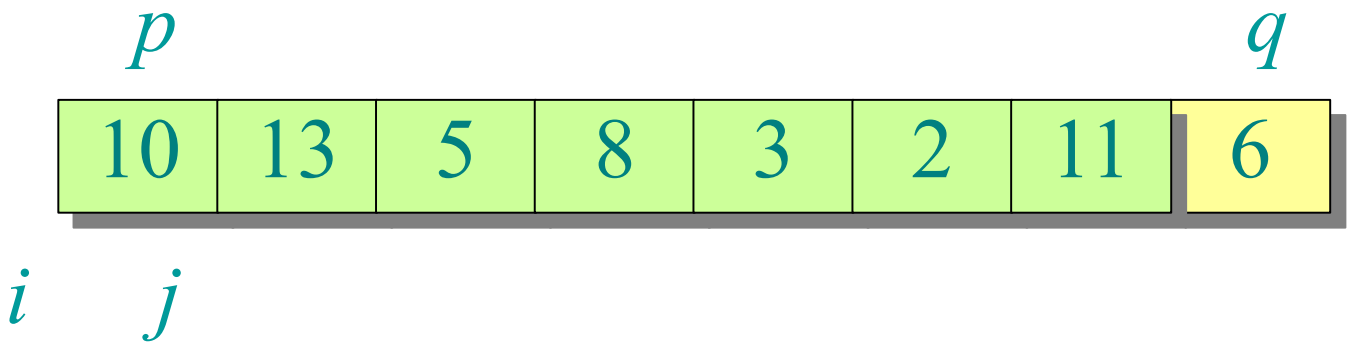
```
PARTITION(A, p, q)   ◁ A[p . . q]
    x ← A[q]         ◁ pivot = A[q]
    i ← p - 1
    for j ← p to q - 1
        do if A[j] ≤ x
                then  i ← i + 1
                      exchange A[i] ↔ A[j]
    exchange A[q] ↔ A[i+1]
    return i
```

Running time $= O(n)$ for n elements.

Invariant:



Example of Partitioning



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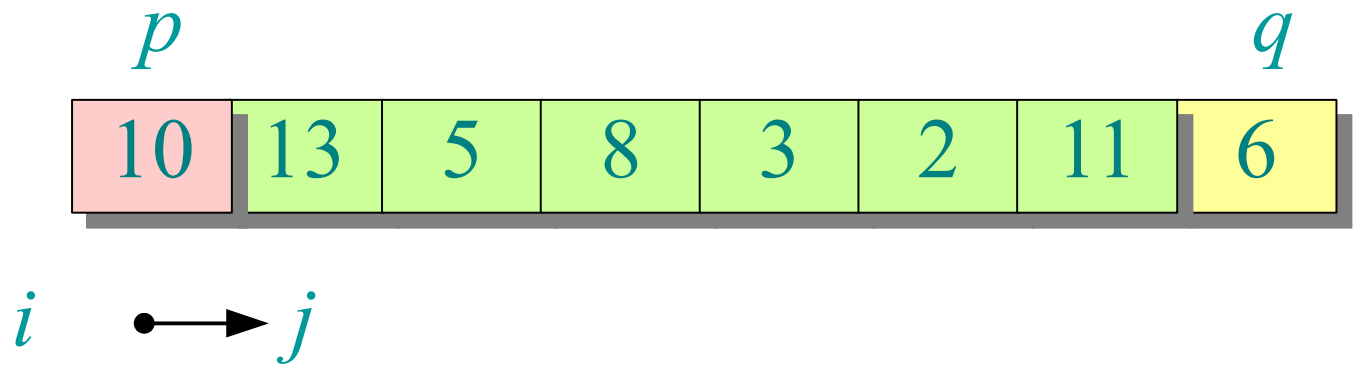

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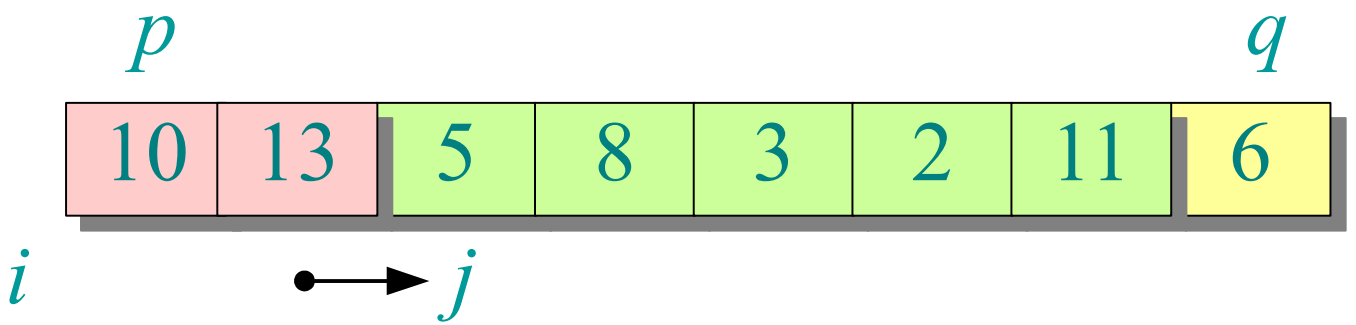

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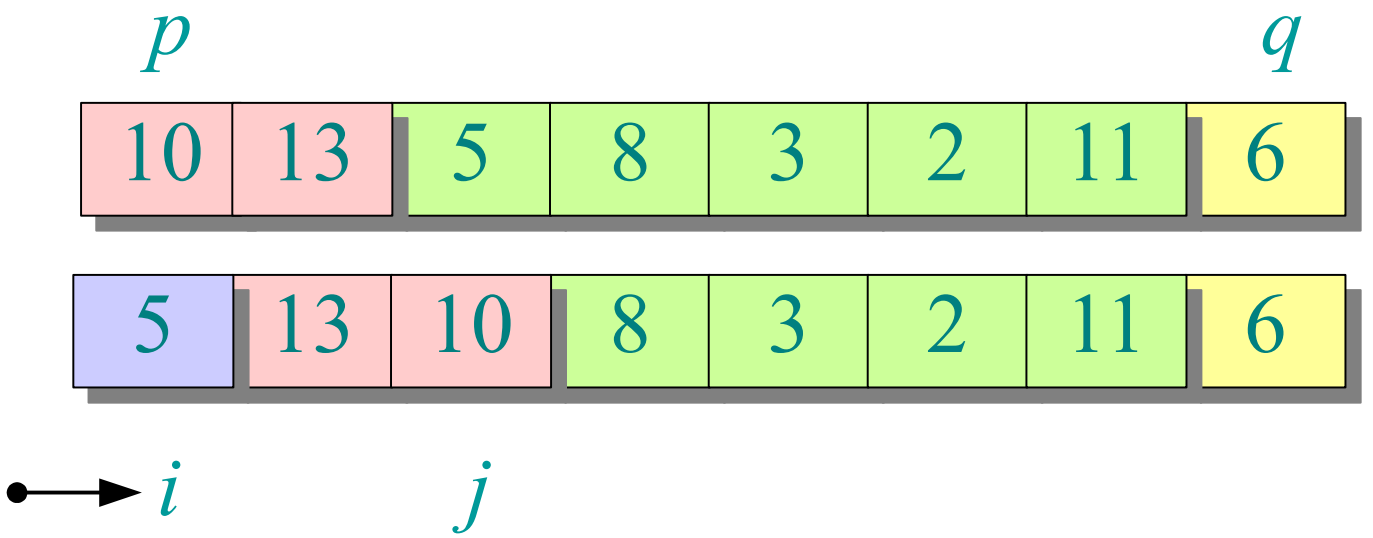

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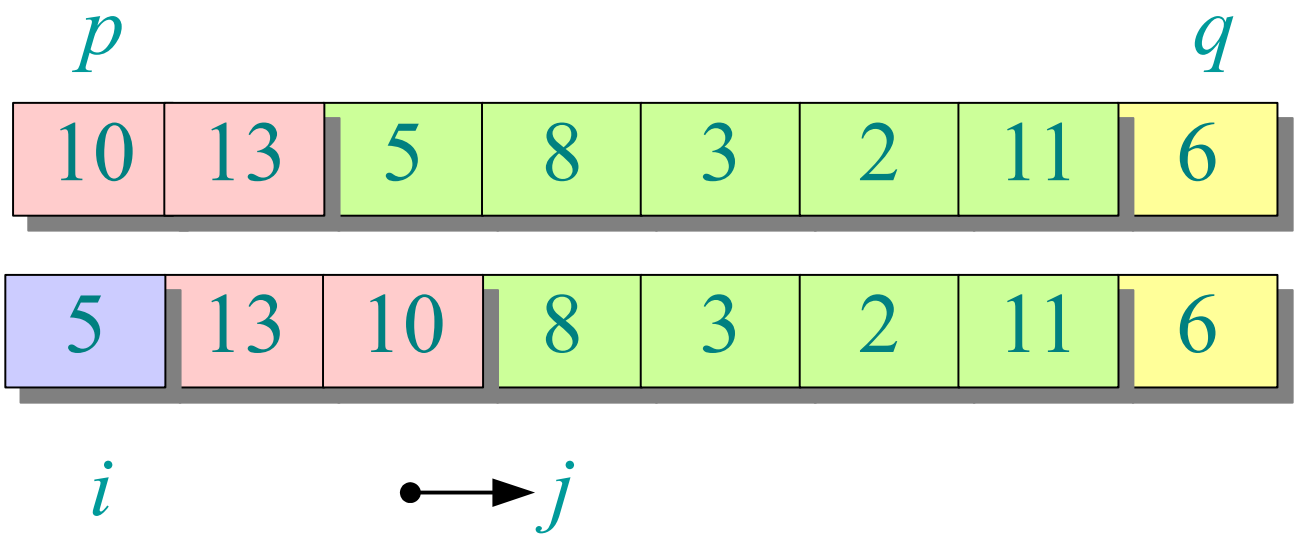

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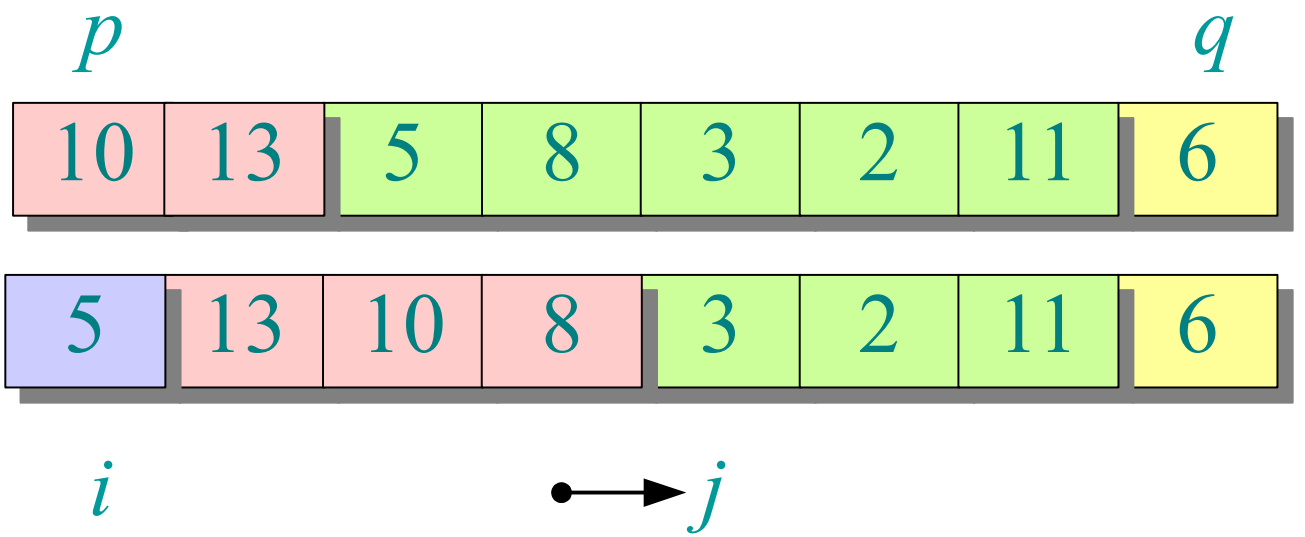

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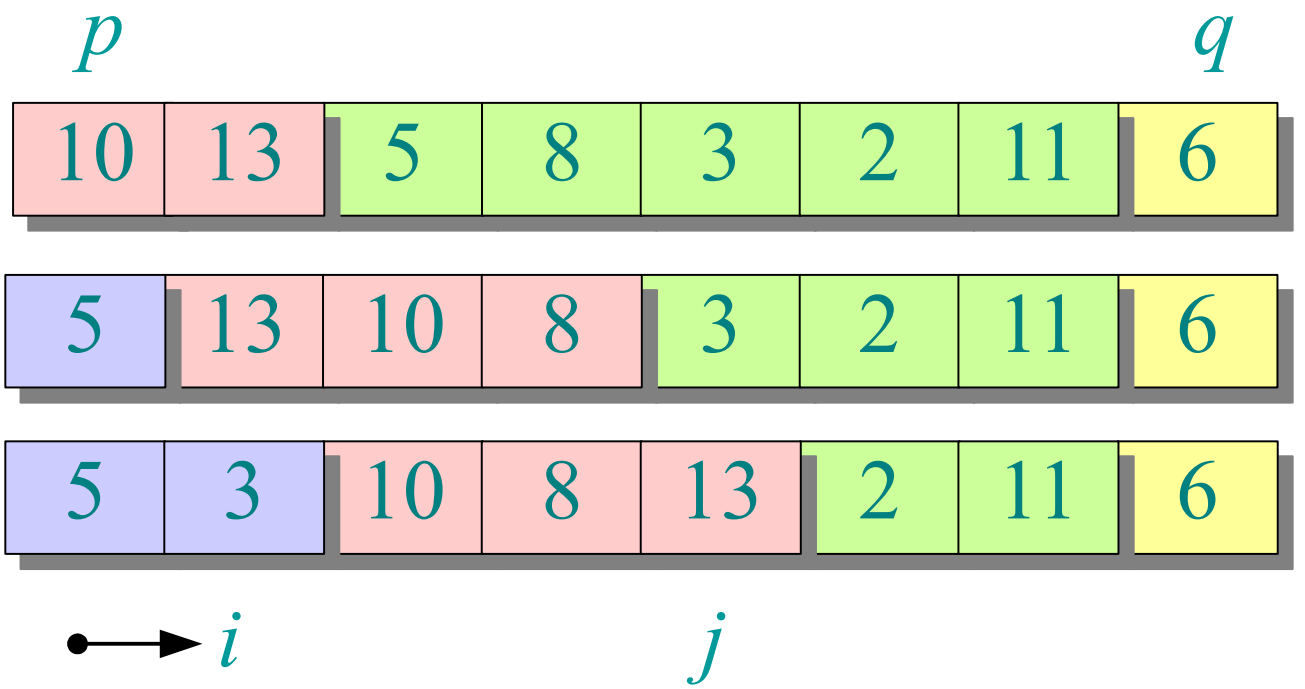

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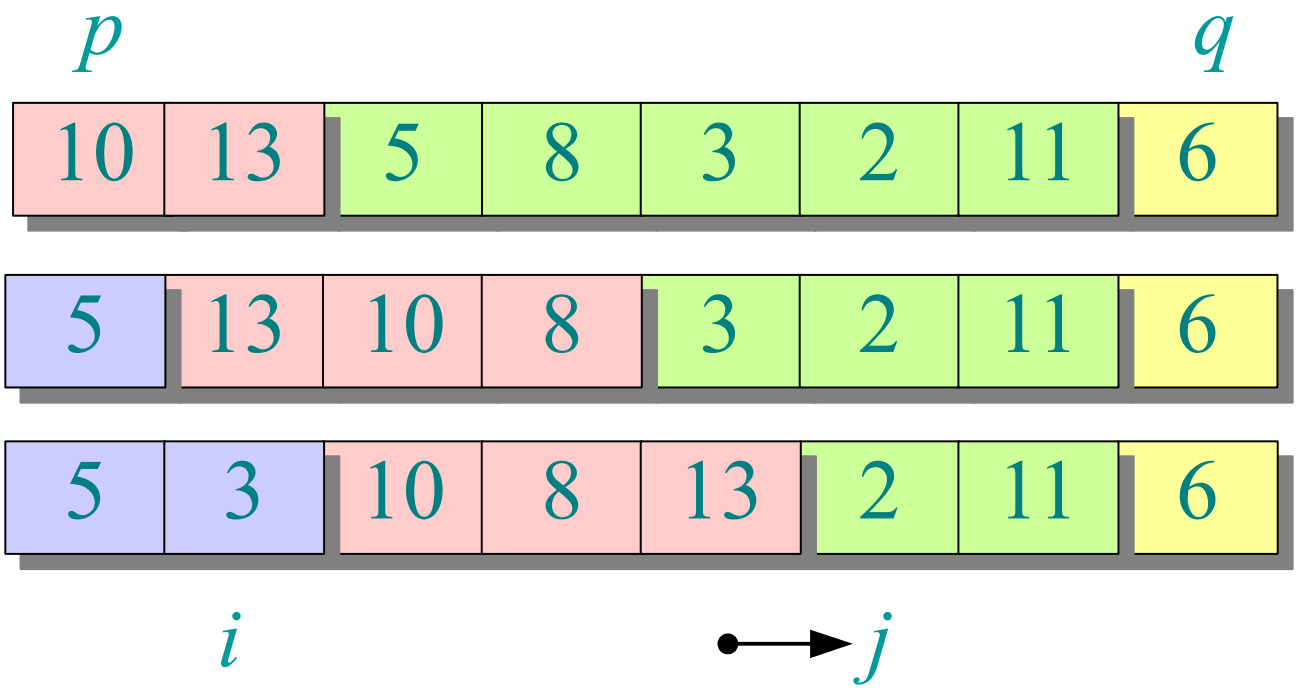

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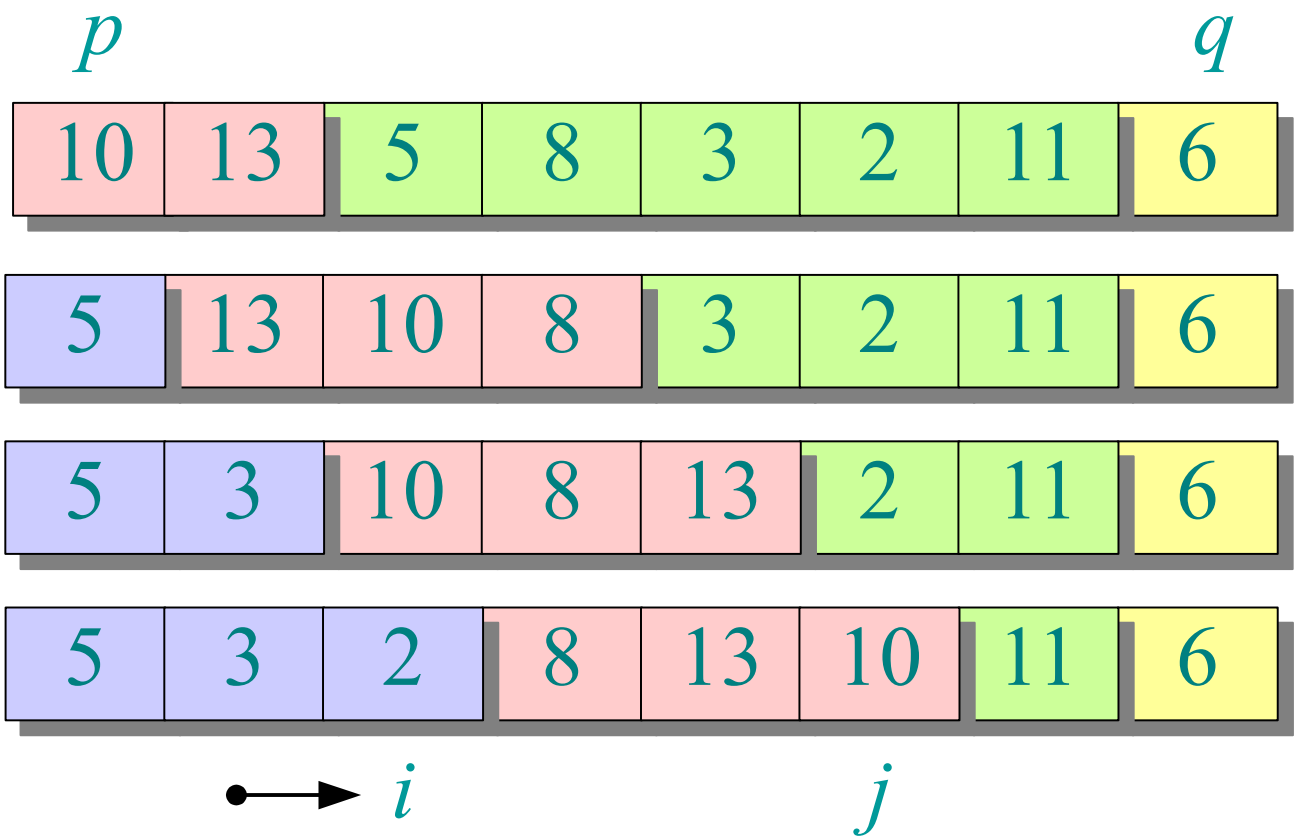

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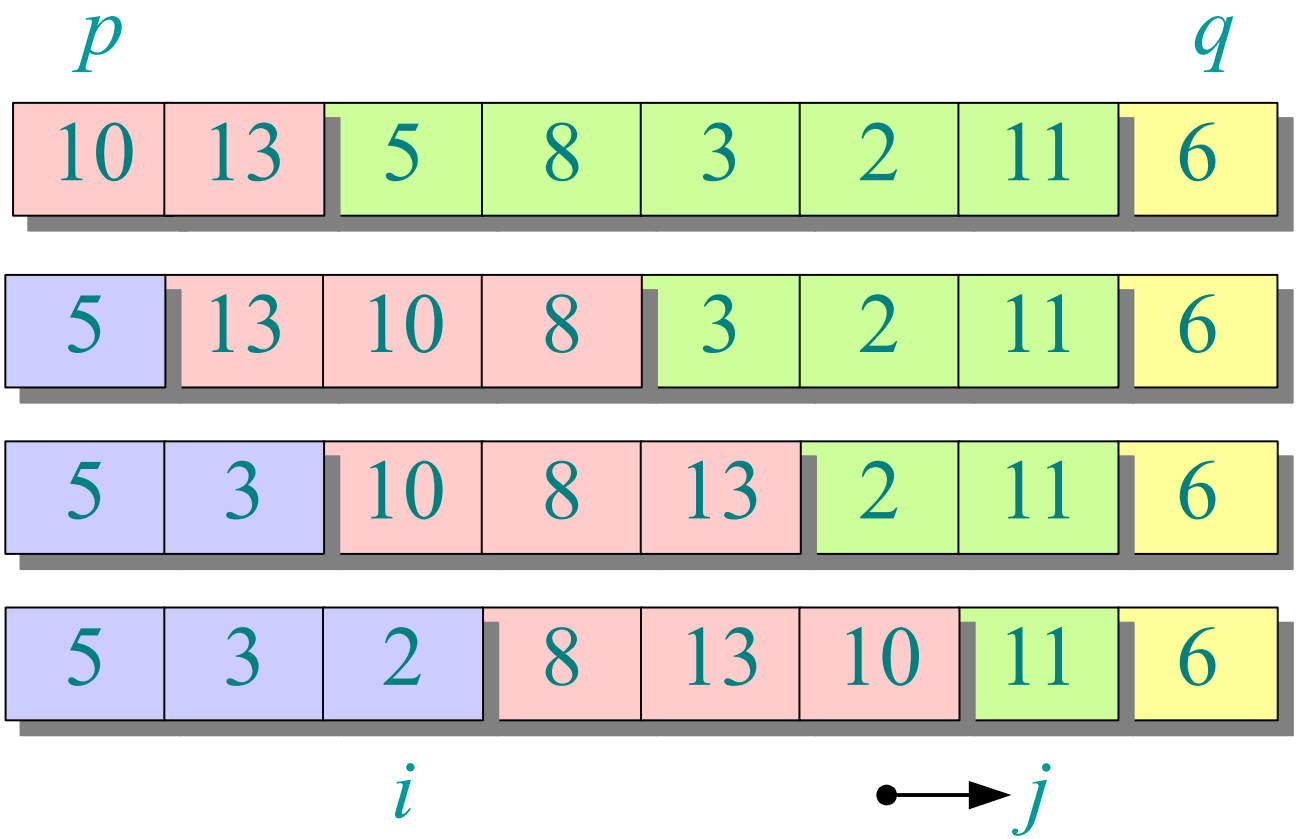

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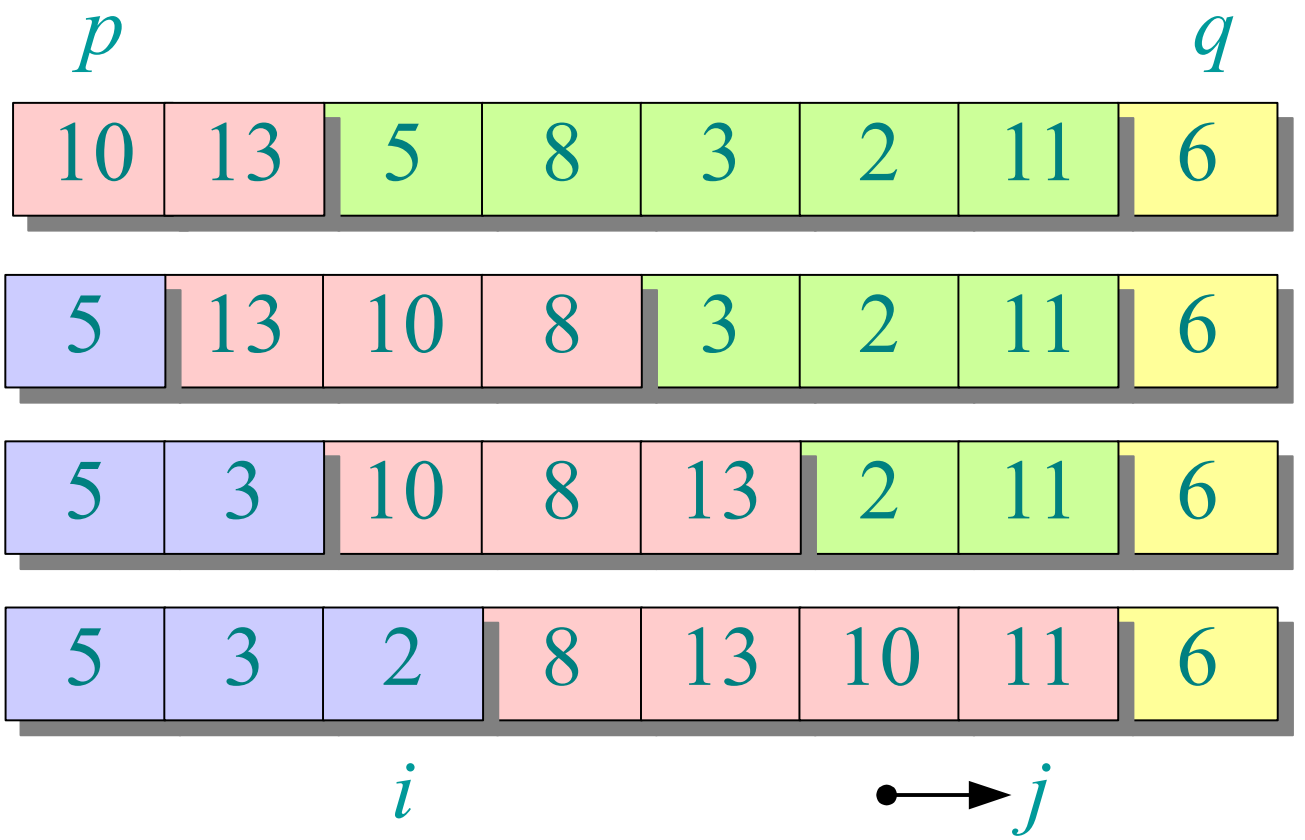

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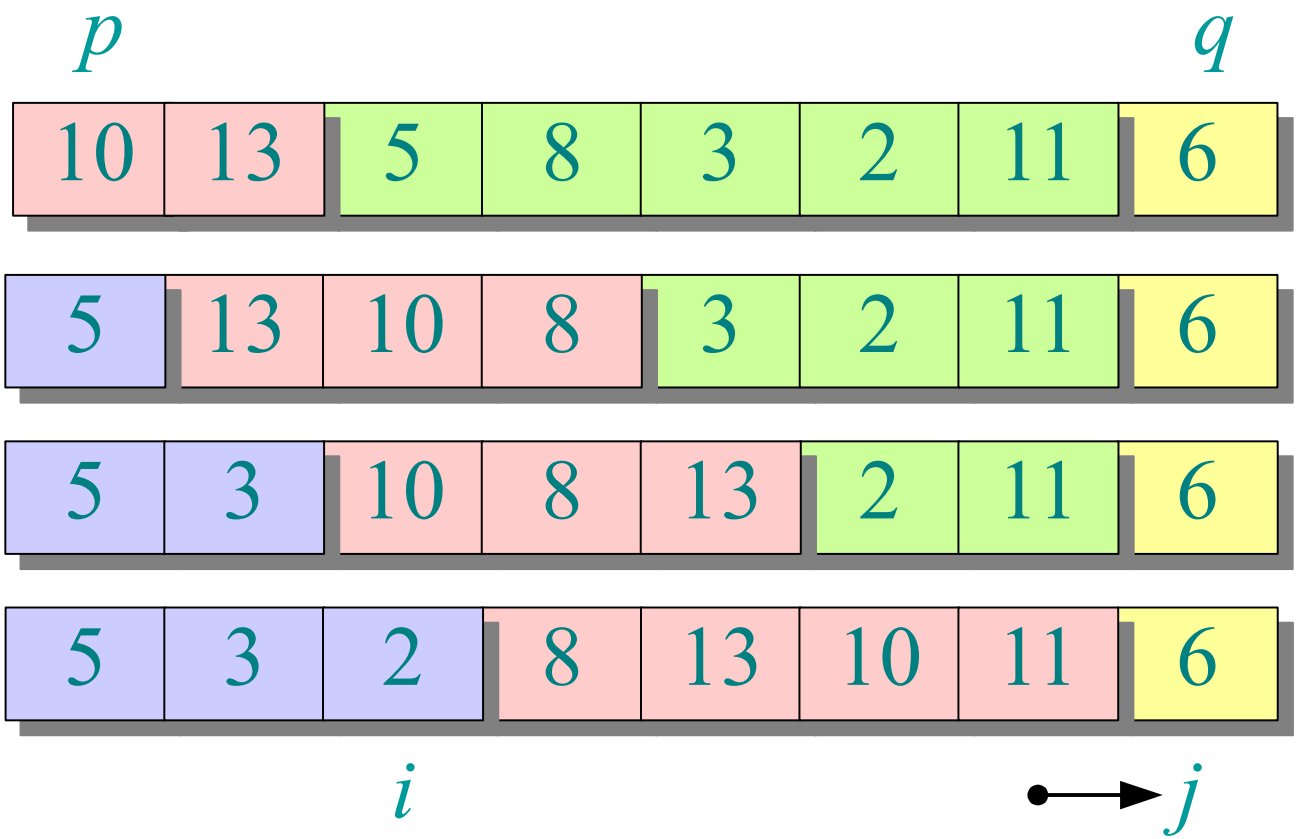

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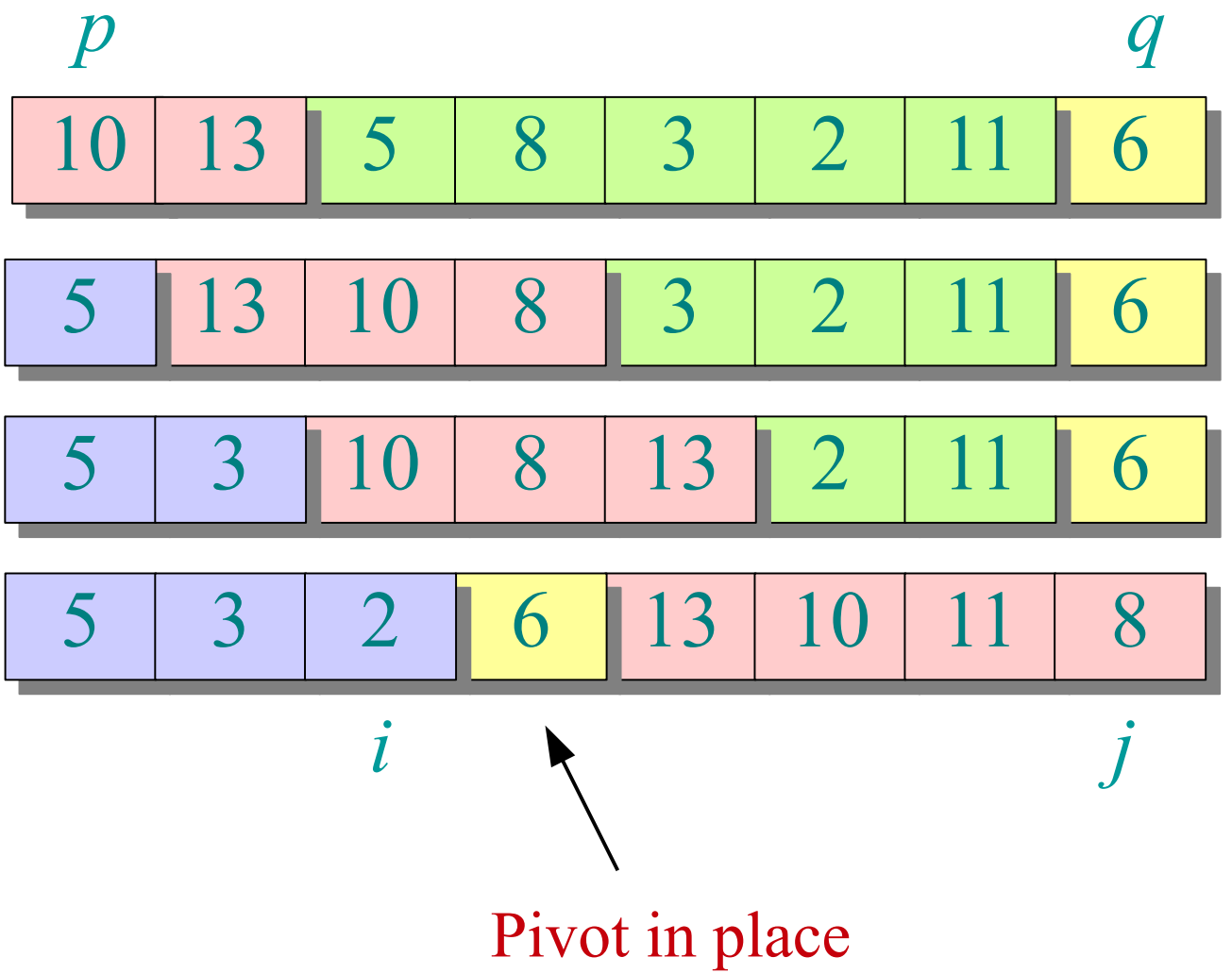

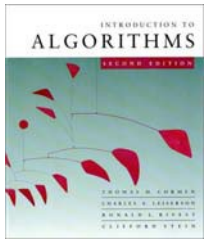

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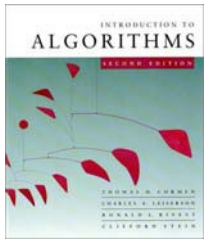
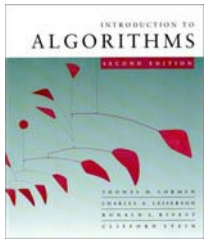


Analysis of quicksort

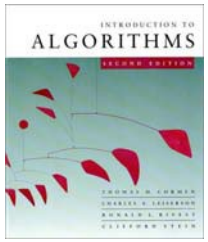
- Assume all input elements are distinct.
- In practice, there are better partitioning algorithms for when duplicate input elements may exist.
- Let $T(n)$ = worst-case running time on an array of n elements.



Worst-case of quicksort

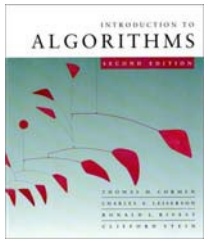
- Input sorted or reverse sorted.
- Partition around min or max element.
- One side of partition always has no elements.

$$
\begin{aligned}
T(n) &= T(0) + T(n-1) + \Theta(n) \\
&= \Theta(1) + T(n-1) + \Theta(n) \\
&= T(n-1) + \Theta(n) \\
&= \Theta(n^2) \qquad \textit{(arithmetic series)}
\end{aligned}
$$



Worst-case recursion tree

$$T(n) = T(0) + T(n-1) + cn$$



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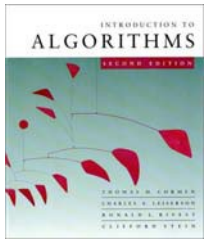
$T(n)$



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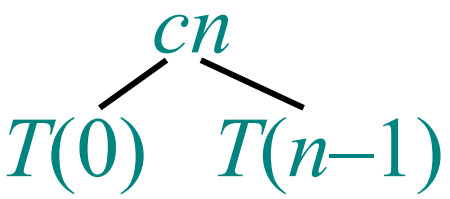
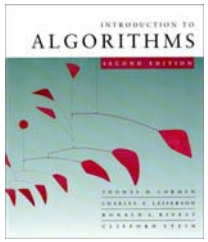
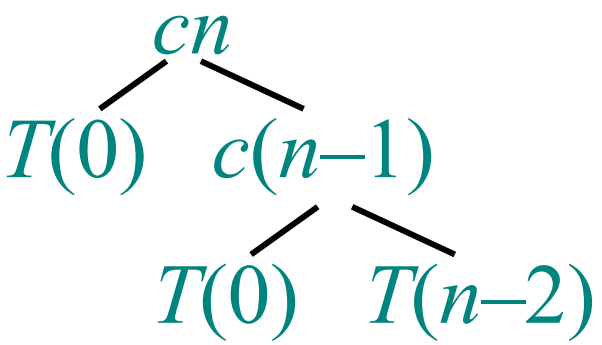
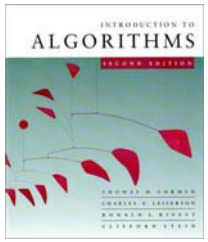




Worst-case recursion tree

$$T(n) = T(0) + T(n–1) + cn$$





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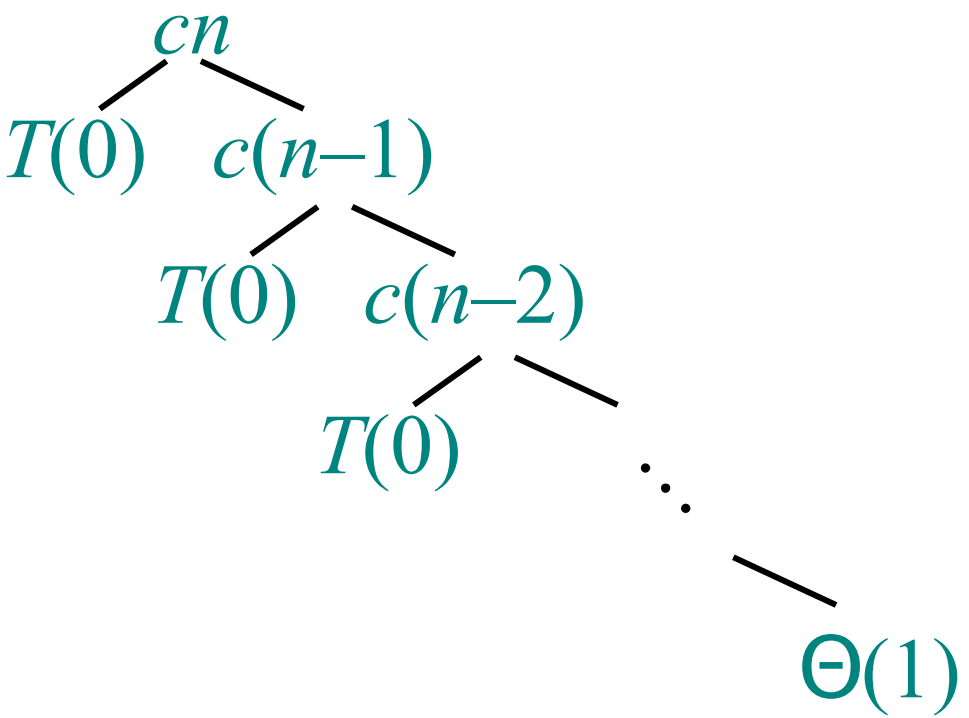

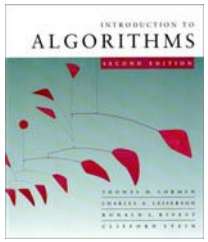

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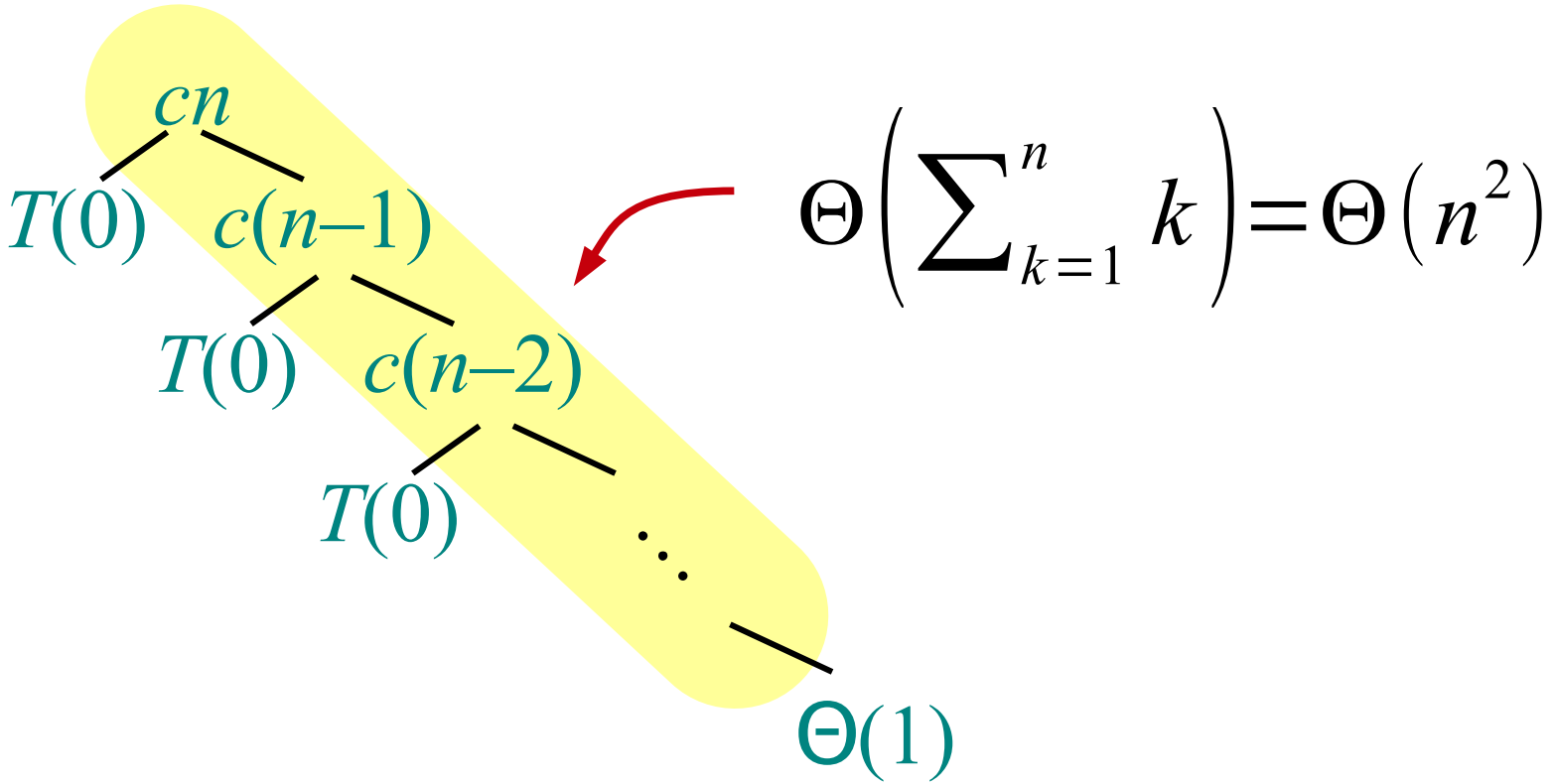
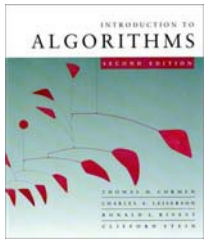


$$\Theta\left(\sum_{k=1}^{n} k\right) = \Theta\left(n^2\right)$$



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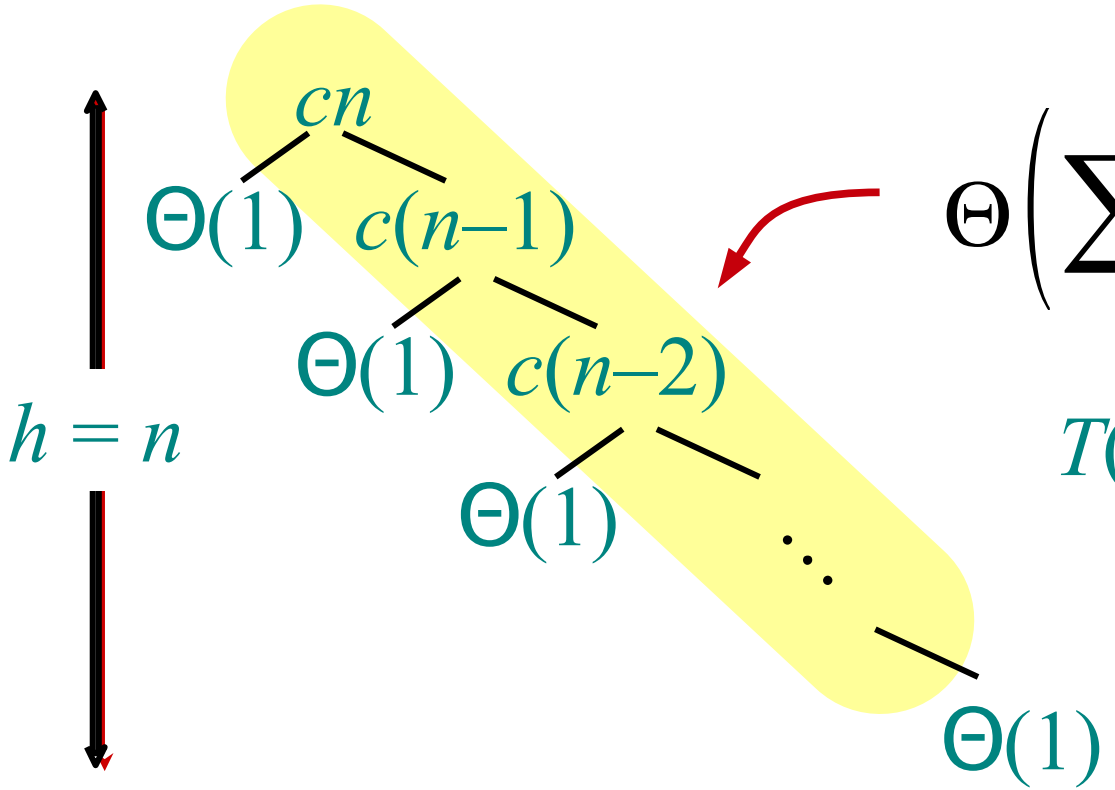
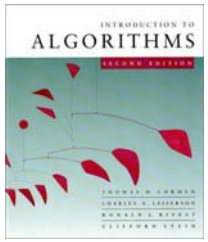


$$\Theta\left(\sum_{k=1}^{n} k\right) = \Theta\left(n^2\right)$$

$$T(n) = \Theta(n) + \Theta(n^2)$$
$$= \Theta(n^2)$$



Best-case analysis

(For intuition only!)

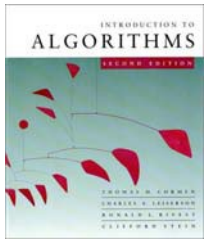
If we're lucky, PARTITION splits the array evenly:

$$T(n) = 2T(n/2) + \Theta(n)$$
$$= \Theta(n \lg n) \qquad \text{(same as merge sort)}$$

What if the split is always $\frac{1}{10} : \frac{9}{10}$?

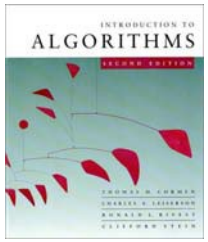
$$T(n) = T\left(\tfrac{1}{10} n\right) + T\left(\tfrac{9}{10} n\right) + \Theta(n)$$

What is the solution to this recurrence?



Analysis of "almost-best" case

$T(n)$



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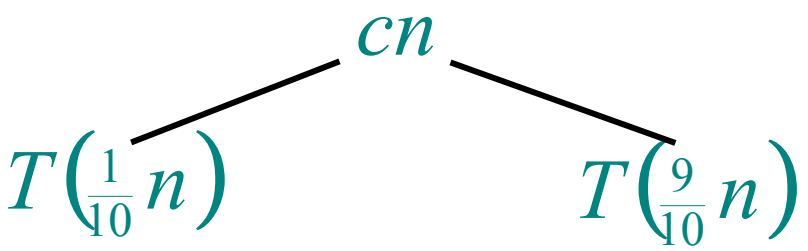
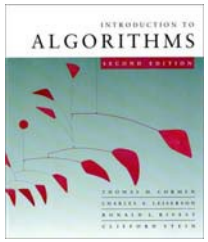
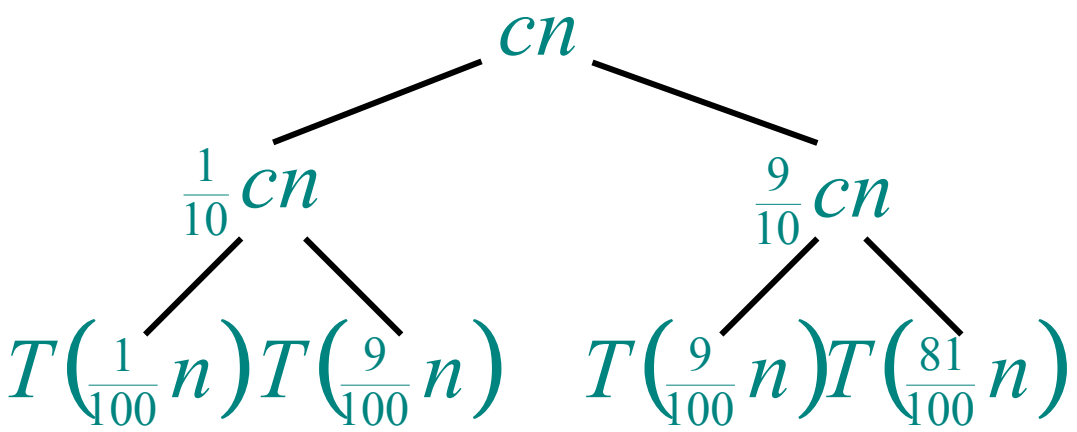
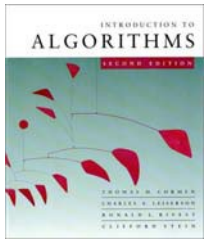


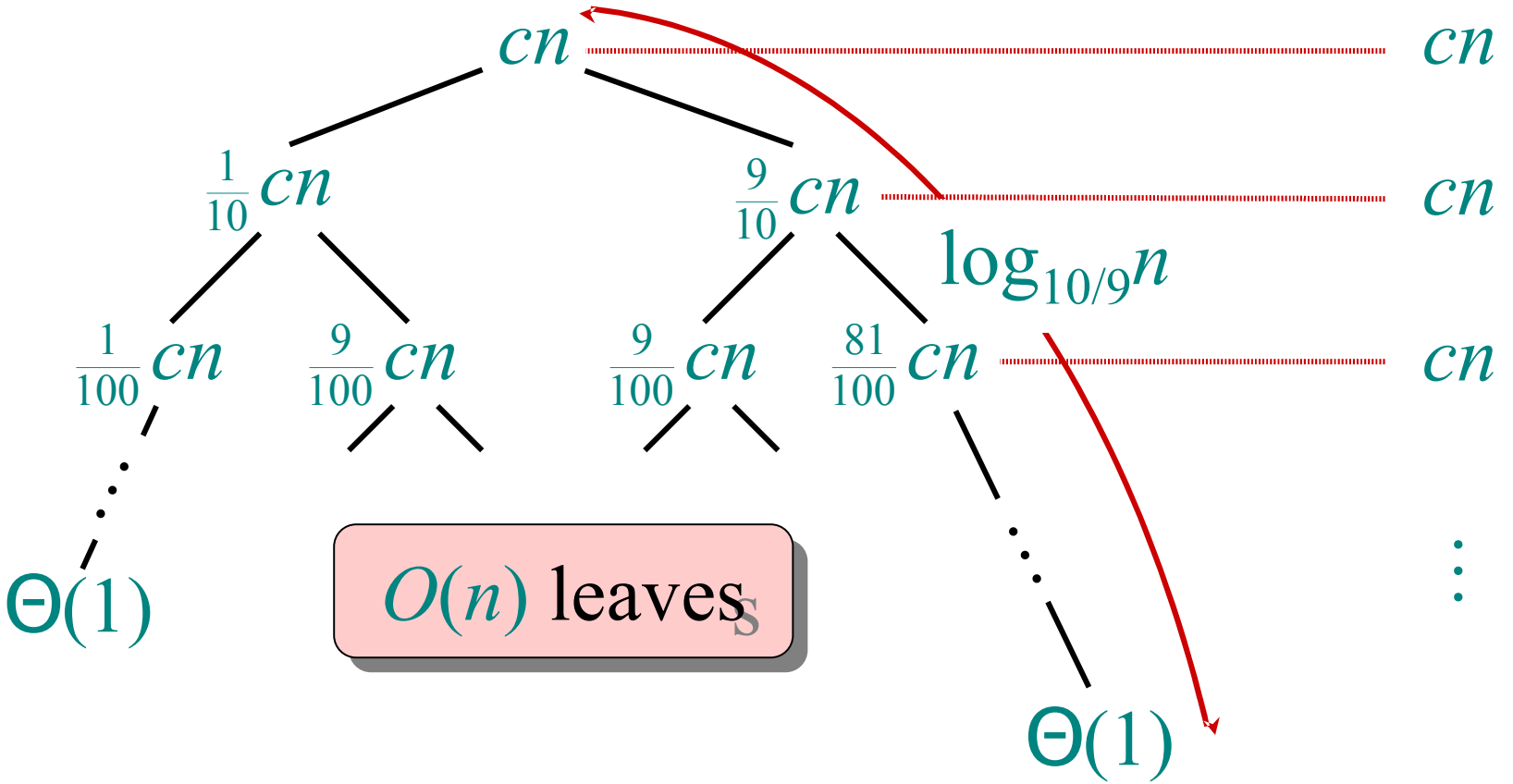


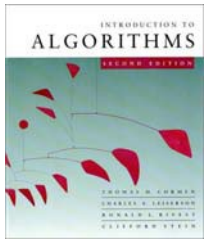
Analysis of "almost-best" case



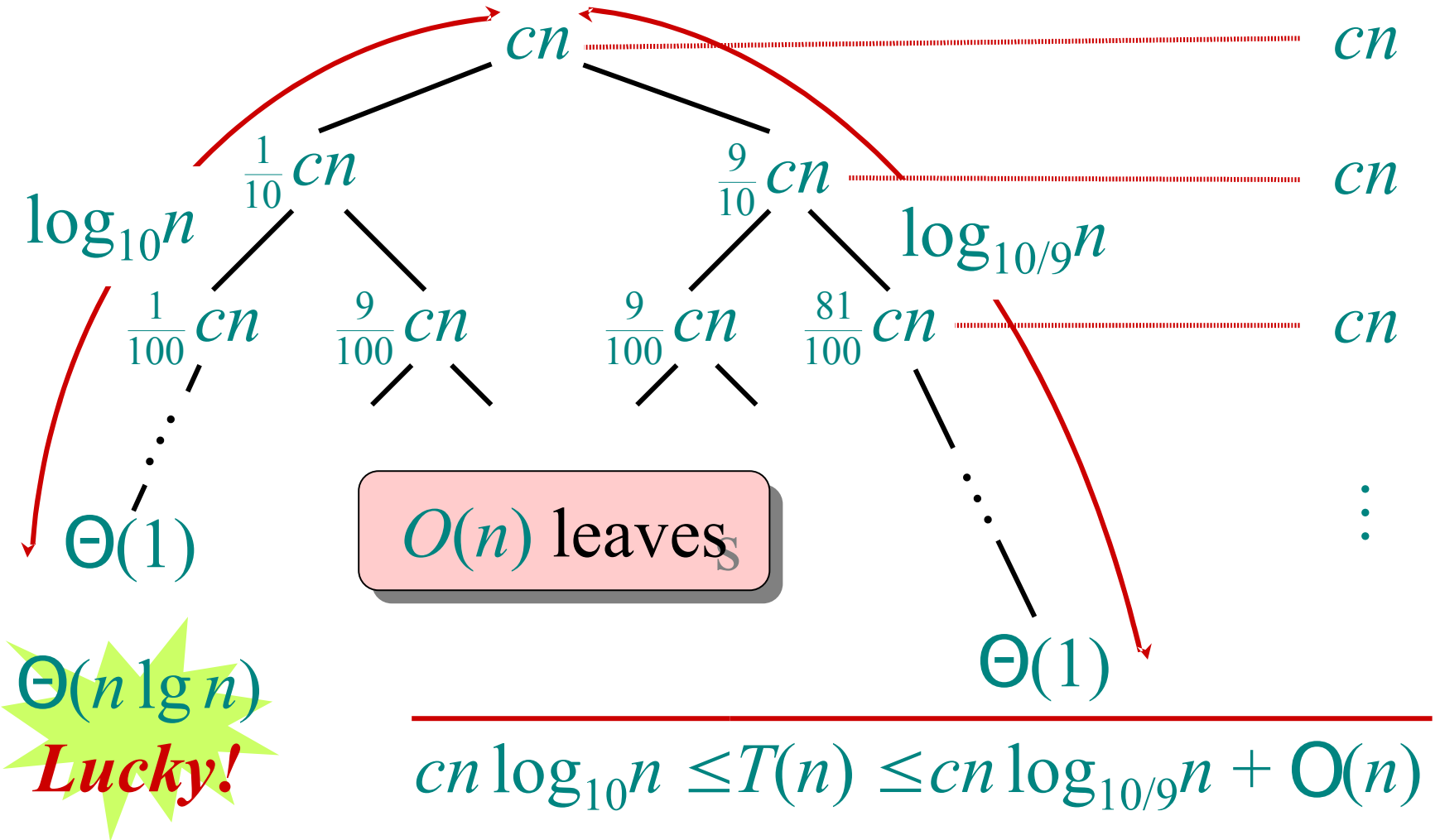


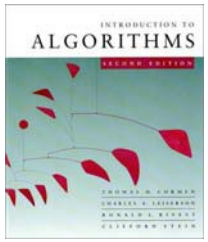
Analysis of "almost-best" case





Analysis of "almost-best" case





More intuition

Suppose we alternate lucky, unlucky, lucky, unlucky, lucky, ….

$L(n) = 2U(n/2) + \Theta(n)$ ***lucky***

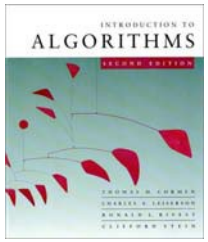
$U(n) = L(n-1) + \Theta(n)$ ***unlucky***

Solving:

$$\begin{aligned} L(n) &= 2(L(n/2-1) + \Theta(n/2)) + \Theta(n) \\ &= 2L(n/2-1) + \Theta(n) \\ &= \Theta(n \lg n) \end{aligned}$$

Lucky!

How can we make sure we are usually lucky?



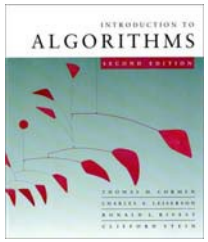
Randomized quicksort

IDEA: Partition around a ***random*** element.

- Running time is independent of the input order.
- No assumptions need to be made about the input distribution.
- No specific input elicits the worst-case behavior.
- The worst case is determined only by the output of a random-number generator.

Randomized Partition

```
RANDOMIZED-PARTITION(A, p, r)
1 i = RANDOM(p, r)
2 exchange A[r] with A[i]
3 return PARTITION(A, p, r)
```



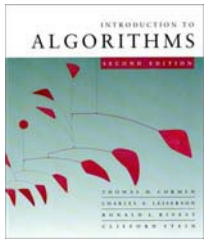
Randomized quicksort analysis

Let $T(n)$ = the random variable for the running time of randomized quicksort on an input of size n, assuming random numbers are independent.

For $k = 0, 1, \ldots, n-1$, define the ***indicator random variable***

$$X_k = \begin{cases} 1 & \text{if PARTITION generates a } k : n-k-1 \text{ split,} \\ 0 & \text{otherwise.} \end{cases}$$

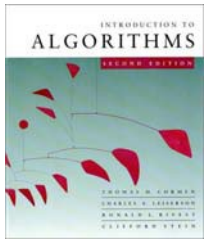
$E[X_k] = \Pr\{X_k = 1\} = 1/n$, since all splits are equally likely, assuming elements are distinct.



Analysis (continued)

$$T(n) = \begin{cases} T(0) + T(n-1) + \Theta(n) & \text{if } 0 : n-1 \text{ split,} \\ T(1) + T(n-2) + \Theta(n) & \text{if } 1 : n-2 \text{ split,} \\ \vdots & \\ T(n-1) + T(0) + \Theta(n) & \text{if } n-1 : 0 \text{ split,} \end{cases}$$

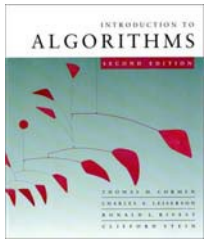
$$= \sum_{k=0}^{n-1} X_k \left(T(k) + T(n-k-1) + \Theta(n) \right)$$



Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k \left(T(k) + T(n-k-1) + \Theta(n)\right)\right]$$

Take expectations of both sides.

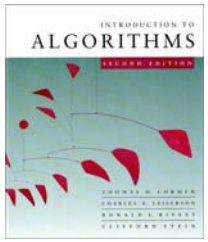


Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k \left(T(k) + T(n-k-1) + \Theta(n)\right)\right]$$

$$= \sum_{k=0}^{n-1} E\left[X_k \left(T(k) + T(n-k-1) + \Theta(n)\right)\right]$$

Linearity of expectation.



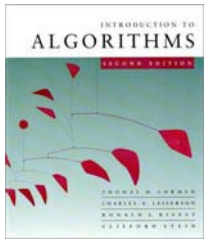
Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k \left(T(k) + T(n-k-1) + \Theta(n)\right)\right]$$

$$= \sum_{k=0}^{n-1} E\left[X_k \left(T(k) + T(n-k-1) + \Theta(n)\right)\right]$$

$$= \sum_{k=0}^{n-1} E[X_k] \cdot E\left[T(k) + T(n-k-1) + \Theta(n)\right]$$

Independence of X_k from other random choices.



Calculating expectation

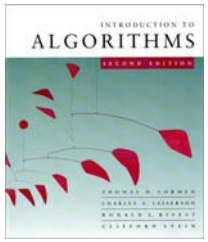
$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k \left(T(k) + T(n-k-1) + \Theta(n)\right)\right]$$

$$= \sum_{k=0}^{n-1} E\left[X_k \left(T(k) + T(n-k-1) + \Theta(n)\right)\right]$$

$$= \sum_{k=0}^{n-1} E[X_k] \cdot E\left[T(k) + T(n-k-1) + \Theta(n)\right]$$

$$= \frac{1}{n}\sum_{k=0}^{n-1} E[T(k)] + \frac{1}{n}\sum_{k=0}^{n-1} E[T(n-k-1)] + \frac{1}{n}\sum_{k=0}^{n-1} \Theta(n)$$

Linearity of expectation; $E[X_k] = 1/n$.



Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k \left(T(k) + T(n-k-1) + \Theta(n)\right)\right]$$

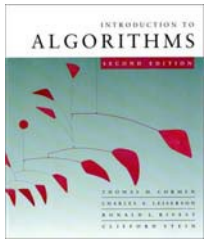
$$= \sum_{k=0}^{n-1} E\left[X_k \left(T(k) + T(n-k-1) + \Theta(n)\right)\right]$$

$$= \sum_{k=0}^{n-1} E[X_k] \cdot E\left[T(k) + T(n-k-1) + \Theta(n)\right]$$

$$= \frac{1}{n}\sum_{k=0}^{n-1} E[T(k)] + \frac{1}{n}\sum_{k=0}^{n-1} E[T(n-k-1)] + \frac{1}{n}\sum_{k=0}^{n-1} \Theta(n)$$

$$= \frac{2}{n}\sum_{k=1}^{n-1} E[T(k)] + \Theta(n)$$

Summations have identical terms.



Hairy recurrence

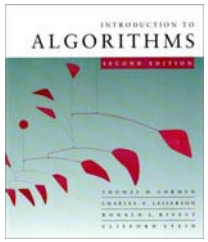
$$E[T(n)] = \frac{2}{n}\sum_{k=2}^{n-1} E[T(k)] + \Theta(n)$$

(The $k = 0, 1$ terms can be absorbed in the $\Theta(n)$.)

Prove: $E[T(n)] \le a\,n \lg n$ for constant $a > 0$.

- Choose a large enough so that $a\,n \lg n$ dominates $E[T(n)]$ for sufficiently small $n \ge 2$.

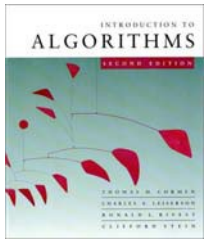
Use fact: $\sum_{k=2}^{n-1} k \lg k \le \frac{1}{2} n^2 \lg n - \frac{1}{8} n^2$ (exercise).



Substitution method

$$E[T(n)] \le \frac{2}{n} \sum_{k=2}^{n-1} ak \lg k + \Theta(n)$$

Substitute inductive hypothesis.

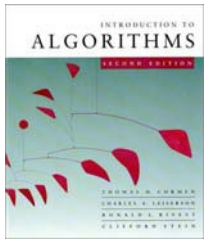


Substitution method

$$E[T(n)] \le \frac{2}{n}\sum_{k=2}^{n-1} ak \lg k + \Theta(n)$$

$$\le \frac{2a}{n}\left[\left(\frac{1}{2}n^2 \lg n - \frac{1}{8}n^2\right)\right] + \Theta(n)$$

Use fact.



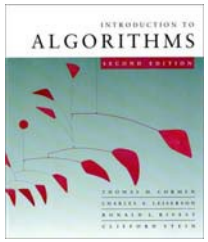
Substitution method

$$E[T(n)] \le \frac{2}{n}\sum_{k=2}^{n-1} ak \lg k + \Theta(n)$$

$$\le \frac{2a}{n}\left(\frac{1}{2}n^2 \lg n - \frac{1}{8}n^2\right) + \Theta(n)$$

$$= an \lg n - \left(\frac{an}{4} - \Theta(n)\right)$$

Express as ***desired – residual***.



Substitution method

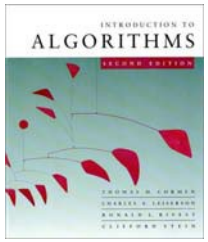
$$E[T(n)] \le \frac{2}{n}\sum_{k=2}^{n-1} ak \lg k + \Theta(n)$$

$$= \frac{2a}{n}\left[\left(\frac{1}{2}n^2 \lg n - \frac{1}{8}n^2\right)\right] + \Theta(n)$$

$$= an \lg n - \left[\left(\frac{an}{4} - \Theta(n)\right)\right]$$

$$\le an \lg n ,$$

if a is chosen large enough so that $an/4$ dominates the $\Theta(n)$.



Quicksort in practice

- Quicksort is a great general-purpose sorting algorithm.
- Quicksort is typically over twice as fast as merge sort.
- Quicksort can benefit substantially from ***code tuning***.
- Quicksort behaves well even with caching and virtual memory.

Summary

- Heaps
  - Useful for building priority queues
- Heapsort
  - Sorts "in place" → $\Theta(1)$ space
  - Worst case time $\Theta(n \lg n)$
- Quicksort
  - Sorts "in place" → $\Theta(1)$ space
  - Best case time $\Theta(n \lg n)$
  - Average case time $\Theta(n \lg n)$
  - Worst case time $\Theta(n^2)$

Recap

- Heapsort
- Quicksort
- Next:
  - Linear time sorting
  - Order statistics