

CMP448: Algorithms



Lecture 09: Greedy Algorithms

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Agenda

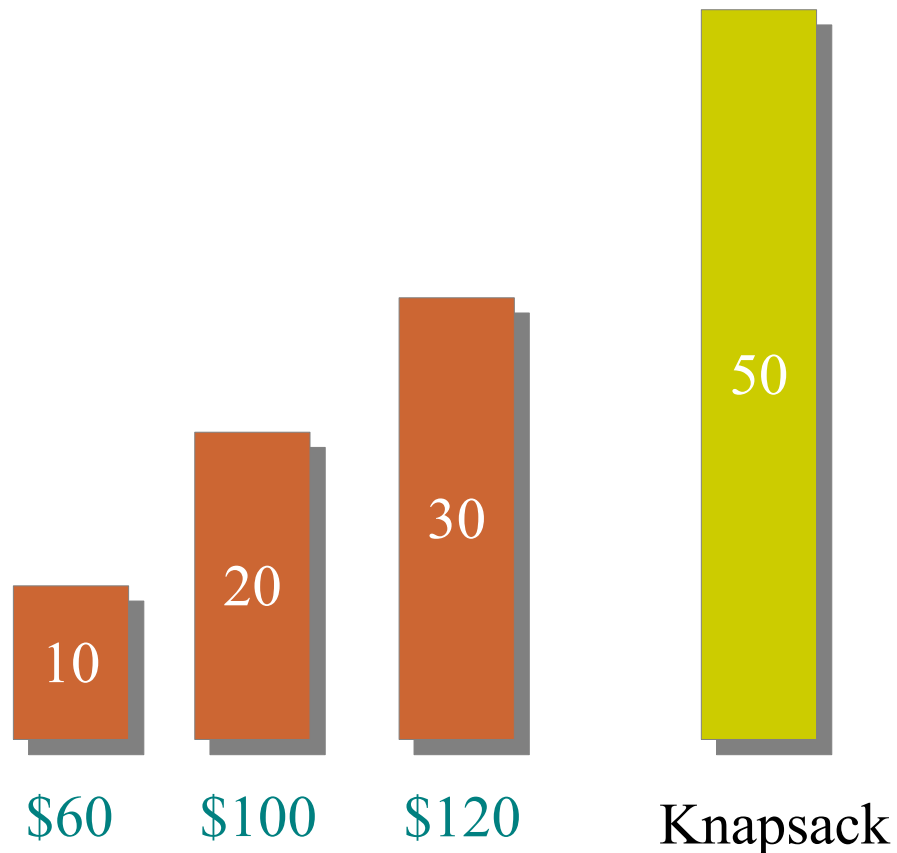
- Knapsack Problem
 - Dynamic Programming Algorithm
 - Greedy Algorithm
- Greedy Algorithm Properties

Acknowledgment

A lot of slides adapted from the slides of Erik Demaine, Charles Leiserson, and David Luebke.

Knapsack Problem

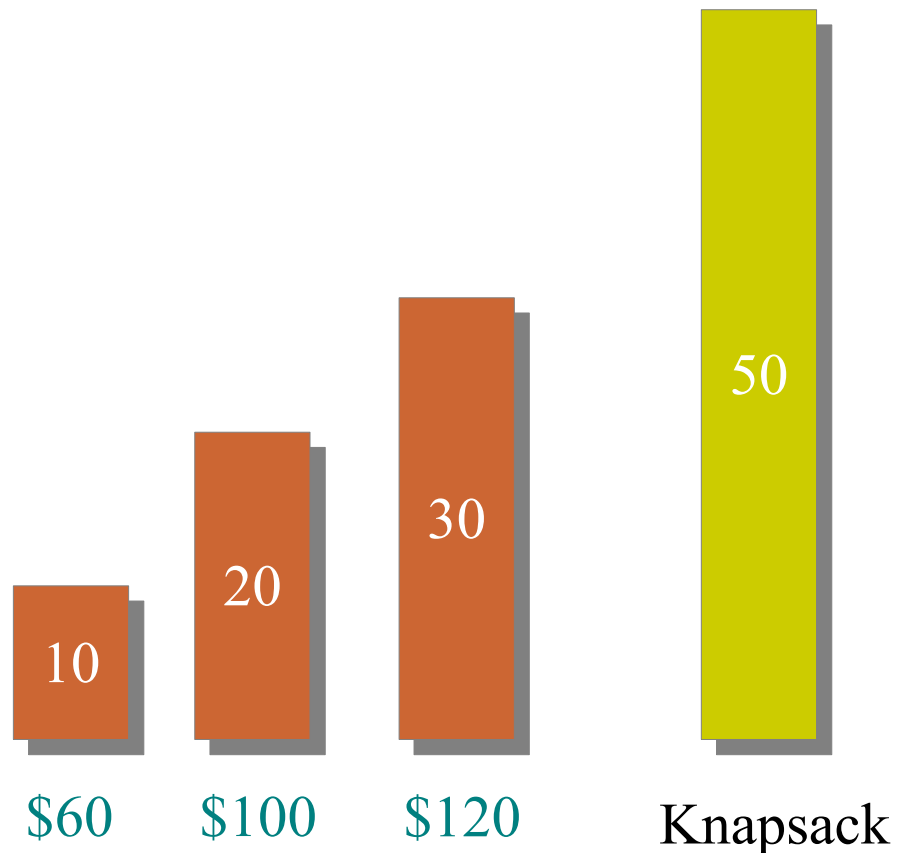
A thief has a **knapsack** of limited **capacity** W , and wants to pack it with items that have as much value as possible such that the total weight does not exceed the knapsack's capacity.



Knapsack Problem

Two types:

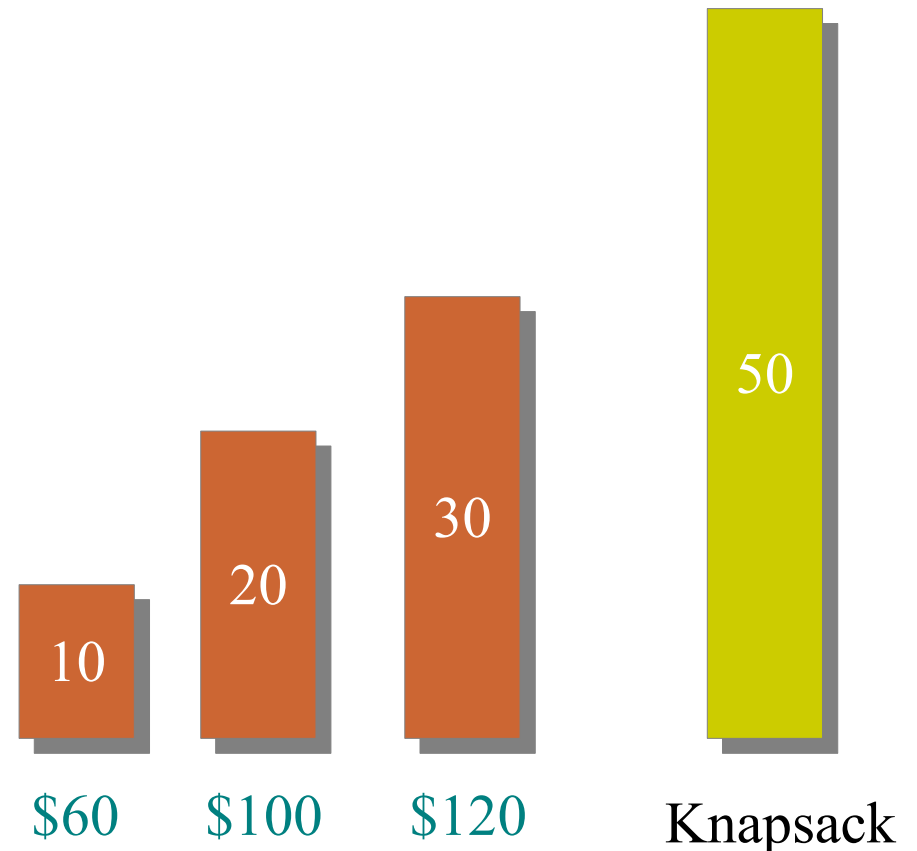
1. **0-1 Knapsack Problem**: can only pack *whole* items
2. **Fractional Knapsack**: can pack *fractions* of items



0-1 Knapsack Problem

Given a knapsack with capacity W and a set S of n items each with weight w_i and value b_i , all integers.

Problem: Find the **subset** of items to put in the knapsack to achieve maximum value.

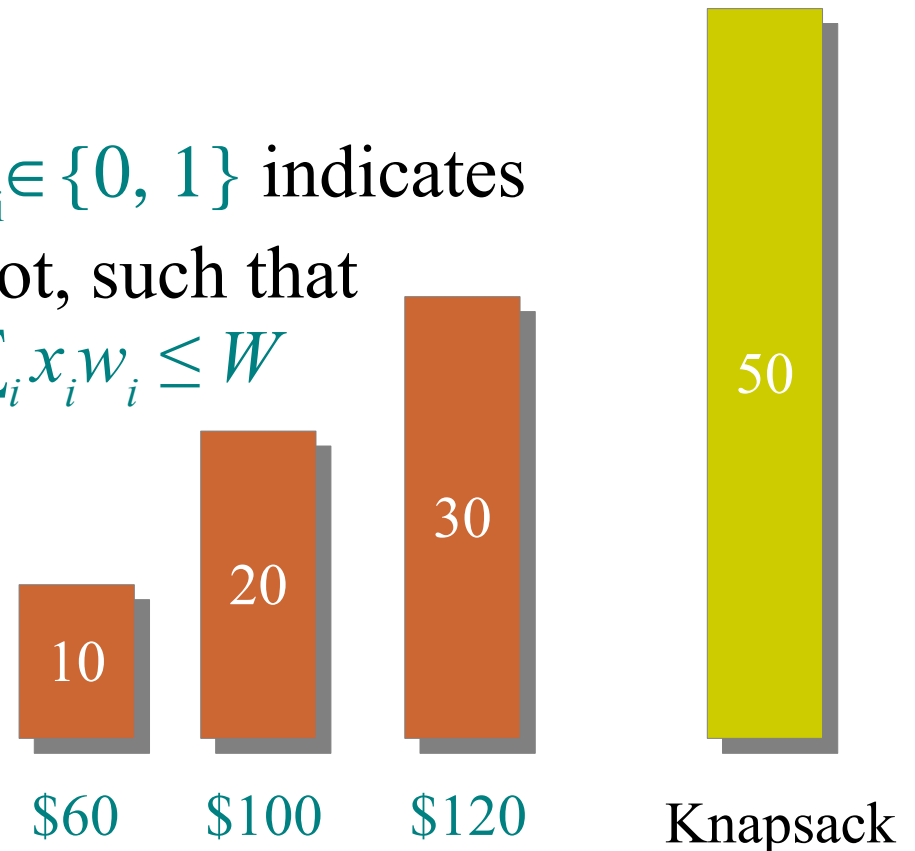


0-1 Knapsack Problem

Given a knapsack with capacity W and a set S of n items each with weight w_i and value b_i , all integers.

Problem: Find the **subset** of items to put in the knapsack to achieve maximum value.

Find $x = [x_1, x_2, \dots, x_n]$ where $x_i \in \{0, 1\}$ indicates whether item i is included or not, such that $x = \operatorname{argmax}_x \sum_i x_i b_i$ **subject to** $\sum_i x_i w_i \leq W$



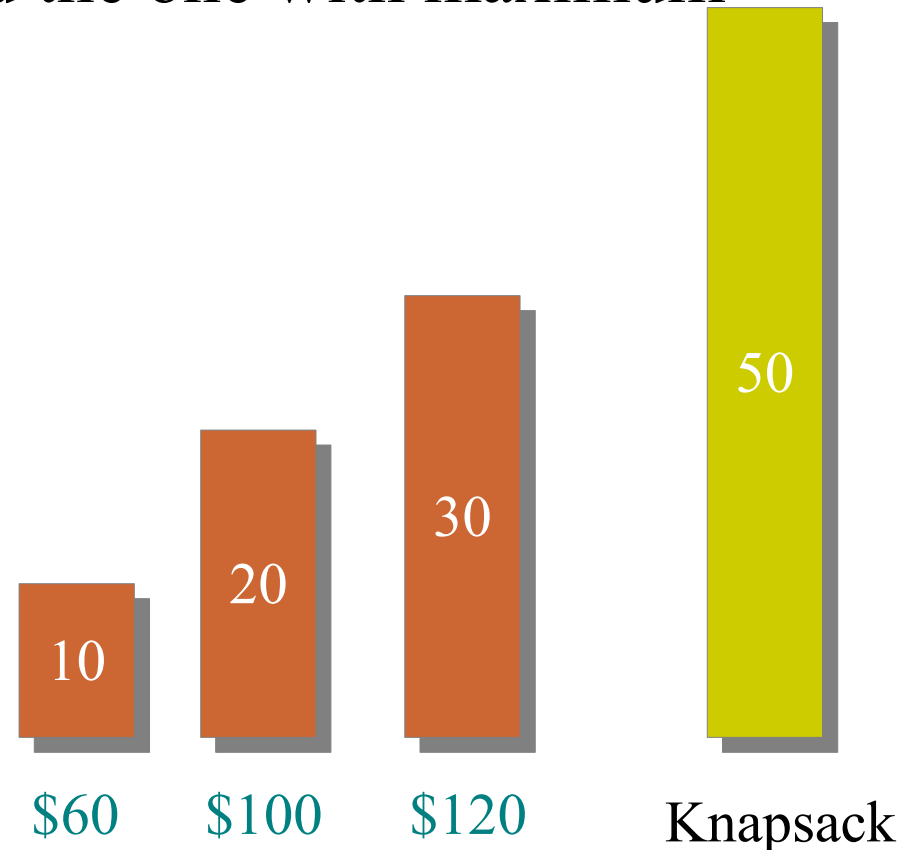
Brute Force Approach

Since we have n items, how many possible subsets?
We have 2^n possible subsets

Algorithm:

Enumerate all subsets, and find the one with maximum value that fits in the knapsack

Takes time $O(2^n)$



Dynamic Programming Approach

Remember, for dynamic programming we need:

1. Optimal substructure

Solution to the problem contains optimal solutions to subproblems

2. Overlapping subproblems

Few distinct subproblems that need to be solved over and over again

Subproblem 1

Define S_k to be the subset of items labeled 1 to k .

The subproblem would be to find the subset of S_k with *maximum* value $B[k]$ that fits in the knapsack.

The solution to the main problem is the solution of S_n .

Question:

Can we describe the solution to the main problem in terms of solutions to subproblems (i.e. **optimal substructure**)?

Answer: No!

Subproblem 1

	1	2	3	4	5
Weight	2	3	4	5	9
Value	3	4	5	8	10

S_4

Knapsack
20

Optimal solution for S_4 contains items $\{1, 2, 3, 4\}$
with *weight* 14 and *value* $B[4] = 20$

Subproblem 1

	1	2	3	4	5
Weight	2	3	4	5	9
Value	3	4	5	8	10

Knapsack

20

S_5

The diagram illustrates a knapsack problem. A table lists five items with their weights and values. A knapsack with a capacity of 20 is shown to the right. A red bracket under the first five items is labeled S_5 , indicating the subproblem of selecting items from the set S_5 .

Optimal solution for S_4 contains items $\{1, 2, 3, 4\}$
with *weight* 14 and *value* $B[4] = 20$

Optimal solution for S_5 contains items $\{1, 3, 4, 5\}$
with *weight* 20 and *value* $B[5] = 26$

Solution to S_5 does **not** contain solution to S_4 !!
Need another definition for the subproblem.

Subproblem 2

Add another parameters w that defines the exact weight of the chosen subset.

Define S_k to be the subset of items labeled 1 to k .

The subproblem would be to find the subset of S_k with *maximum* value that fits into a knapsack with weight w .

Define $B[k, w]$ to be the maximum value attained by a subset in S_k with weight $\leq w$

The solution to the main problem is $B[n, W]$.

Recursive Solution

Assume you solved the problem for $B[k - 1, w]$ and want check if item k should be added to the optimal solution:

$$B[k, w] = \begin{cases} B[k - 1, w] & \text{if } w_k > w \\ \max(B[k - 1, w], B[k - 1, w - w_k] + b_k) & \text{otherwise} \end{cases}$$

In other words, the optimal solution to $B[k, w]$ is either:

1. $B[k - 1, w]$ if item k does **not** fit in the knapsack with capacity w and then can **not** be part of the solution
2. The best of $B[k - 1, w]$ (without item k) or the value obtained by adding item k to $B[k - 1, w - w_k]$

Proof: **exercise!**

Recursive Solution

Top-down Solution

Knapsack(B, k, w)

if $w[k] > w$

$B[k, w] = \text{Knapsack}(B, k-1, w)$

else

$B[k, w] = \max \{ \text{Knapsack}(B, k-1, w),$
 $\text{Knapsack}(B, k-1, w-w[k]) + b[k] \}$

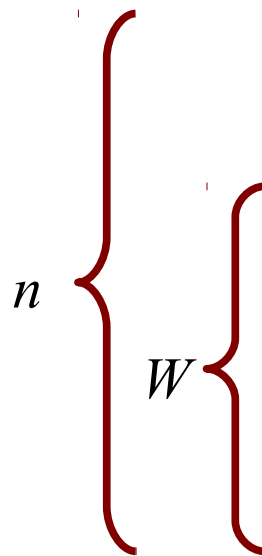
Dynamic Programming Solution

Bottom-up Solution

```
Knapsack( $B, n, W$ )
  for  $w = 0$  to  $W$ 
     $B[0, w] = 0$ 
  for  $k = 1$  to  $n$ 
     $B[k, 0] = 0$ 
    for  $w = 0$  to  $W$ 
      if  $w[k] > w$ 
         $B[k, w] = B[k-1, w]$ 
      else
         $B[k, w] = \max(B[k-1, w], B[k-1, w-w[k]] + b[k])$ 
```

Dynamic Programming Solution

Bottom-up Solution



Knapsack(B, n, W)
for $w = 0$ to W
 $B[0, w] = 0$
for $k = 1$ to n
 $B[k, 0] = 0$
 for $w = 0$ to W
 if $w[k] > w$
 $B[k, w] = B[k-1, w]$
 else
 $B[k, w] = \max(B[k-1, w], B[k-1, w-w[k]] + b[k])$

Running time? $\Theta(n W)$

0-1 Knapsack Summary

Brute-force solution: takes time $O(2^n)$

Dynamic Programming solution takes time $O(n W)$

What about the **Fractional Knapsack** problem?

Will solve it using **Greedy Algorithms**, a design strategy related to Dynamic Programming, that makes **locally optimal** (*greedy*) choices at each step hoping to arrive at a **global** optimum.

It works for some problems i.e. finds the global optimum, and not others!

Greedy Algorithms

Related to Dynamic Programming, depends on two properties:

1. Optimal substructure

The optimal solution to **the main problem** contains optimal solutions to **subproblems** (**like** DP).

2. Greedy choice

At each step, we only need to solve **one subproblem** and make one choice, the **greedy choice** i.e. the best choice at the moment (**unlike** DP).

Finding the locally optimal solutions will get us to the global optimum.

Greedy Solution for 0-1 Knapsack

Intuition: We want to pack the knapsack with as much value as possible. So what would be our *greedy* choice?

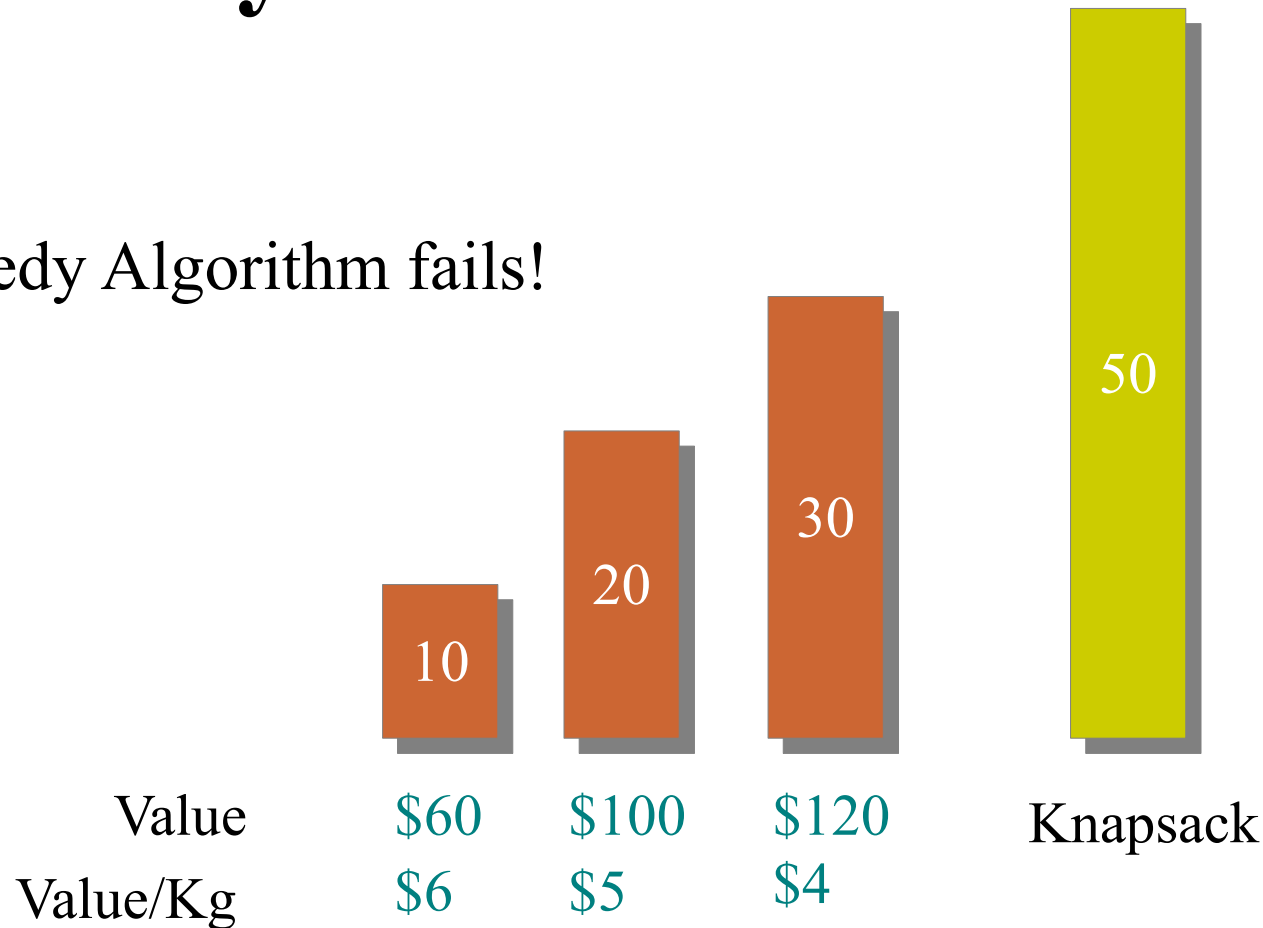
Choose the item with *maximum* value per Kg.

Does it find the *global optimum*?

No!

Greedy Solution for 0-1 Knapsack

Greedy Algorithm fails!



Item 1 has the best value/Kg.

Greedy solution (items 1 & 2) = \$70 < the optimum value of \$220!

Greedy Solution for Fractional Knapsack

Define $b[k]$ to be the **value per Kg** for item k . The problem is to find fractional weights $x[k]$ for each item to pack the knapsack such that the total weight does not exceed W .

Sort the items by their $b[k]$ in increasing order. Define $B[k, w]$ as the **maximum** value obtained from the first k items that can fit a knapsack of weight w .

We get:

$$B[k, w] = x[k] b[k] + B[k-1, w - x[k]]$$
$$x[k] = \min(w[k], w)$$

Amount we are taking from item k

Solution of the **subproblem**

Greedy choice = **Maximum** amount we can take from item k

Greedy Solution for Fractional Knapsack

$$B[k, w] = x[k] b[k] + B[k-1, w - x[k]]$$
$$x[k] = \min(w[k], w)$$

Optimal Substructure

If $B[k, w]$ is the optimal solution for k and w , then $B[k-1, w-x[k]]$ is the optimal solution for $k-1$ and $w-x[k]$.

Proof: (Cut and paste argument)

Assume that $B[k-1, w-x[k]]$ is not optimal.

Then there exists another choice of $x[i]$ for $i = 1 \dots k-1$ that provides bigger value. Therefore we can substitute these choices and add to them $x[k]$ to find a value better than $B[k, w]$, which is a **contradiction**.

Greedy Solution for Fractional Knapsack

$$B[k, w] = x[k] b[k] + B[k-1, w - x[k]]$$
$$x[k] = \min(w[k], w)$$

Greedy Choice Property

If $B[k, w]$ is the optimal solution for k and w , then the greedy choice $x[k] = \min(w[k], w)$ where $b[k]$ is largest value per Kg is part of the the optimal solution.

Proof: (Cut and paste argument)

Assume that $x[k] < \min(w[k], w)$. Since $b[k] > b[i] \forall i = 1 .. k-1$, we can reduce $x[k-1]$ and increase $x[k]$ by the same amount to get a larger $B[k, w]$, which is a **contradiction**.

Recursive Solution for Fractional Knapsack

Recursive Solution

FKnapsack(k, w)

$$x[k] = \min(w[k], w)$$

$$B[k, w] = x[k] b[k] + \text{FKnapsack}(k-1, w - x[k])$$

Running time?

$$\Theta(n \lg n)$$

Time to sort + recursive calls.

Recursive Solution for Fractional Knapsack

Iterative Solution

```
FKnapsack( $n, W$ )  
   $B = 0$   
  for  $k = n$  downto 1  
     $x[k] = \min(w[k], W)$   
     $B += x[k] b[k]$   
     $W -= x[k]$ 
```

Fractional Knapsack Summary

Can solve with greedy algorithms in $\Theta(n \lg n)$

Two properties of Greedy Algorithms:

1. Optimal Substructure
2. Greedy Choice (one subproblem)

Next: Minimal Spanning Trees

Recap

- Knapsack Problem
 - Dynamic Programming Algorithm
 - Greedy Algorithm
- Greedy Algorithm Properties
- Next:
 - Minimal Spanning Trees