

CMP448: Algorithms



Lecture 15: All-Pairs Shortest Paths

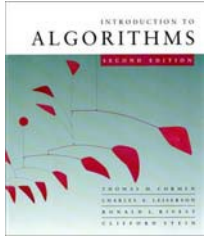
Mohamed Alaa El-Dien Aly
Computer Engineering Department
Cairo University
Spring 2013

Agenda

- Matrix Multiplication Algorithm
- Floyd-Warshall Algorithm
- Johnson's Algorithm

Acknowledgment

A lot of slides adapted from the slides of Erik Demaine and Charles Leiserson.



Shortest paths

Single-source shortest paths

- Nonnegative edge weights

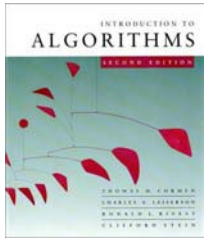
Dijkstra's algorithm: $O(E \lg V)$

- General

Bellman-Ford algorithm: $O(VE)$

- DAG

One pass of Bellman-Ford: $O(V + E)$



Shortest paths

Single-source shortest paths

- Nonnegative edge weights

Dijkstra's algorithm $O(E \lg V)$

- General

Bellman-Ford: $O(VE)$

- DAG

One pass of Bellman-Ford: $O(V + E)$

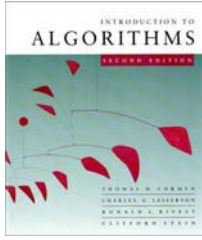
All-pairs shortest paths

- Nonnegative edge weights

Dijkstra's algorithm $|V|$ times: $O(VE \lg V)$

- General

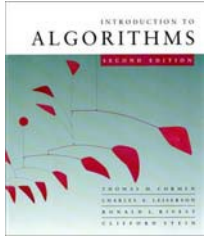
Three algorithms today.



All-pairs shortest paths

Input: Digraph $G = (V, E)$, where $V = \{1, 2, \dots, n\}$, with edge-weight function $w : E \rightarrow \mathbb{R}$.

Output: $n \times n$ matrix of shortest-path lengths $\delta(i, j)$ for all $i, j \in V$.



All-pairs shortest paths

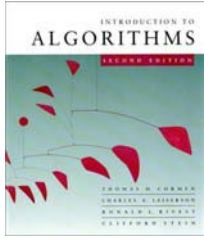
Input: Digraph $G = (V, E)$, where $V = \{1, 2, \dots, n\}$, with edge-weight function $w : E \rightarrow \mathbb{R}$.

Output: $n \times n$ matrix of shortest-path lengths $\delta(i, j)$ for all $i, j \in V$.

IDEA:

- Run Bellman-Ford once from each vertex.
- Time = $O(V^2E)$.
- Dense graph (n^2 edges) $\Rightarrow \Theta(n^4)$ time in the worst case.

Good first try!



Dynamic programming

Consider the $n \times n$ adjacency matrix $A = (a_{ij})$ of the digraph, and define

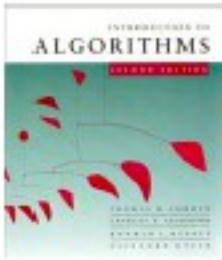
$d_{ij}^{(m)}$ = weight of a shortest path from i to j that uses at most m edges.

Claim: We have

$$d_{ij}^{(0)} = \begin{cases} 0 & \text{if } i = j, \\ \infty & \text{if } i \neq j; \end{cases}$$

and for $m = 1, 2, \dots, n - 1$,

$$d_{ij}^{(m)} = \min_k \{ d_{ik}^{(m-1)} + a_{kj} \}.$$

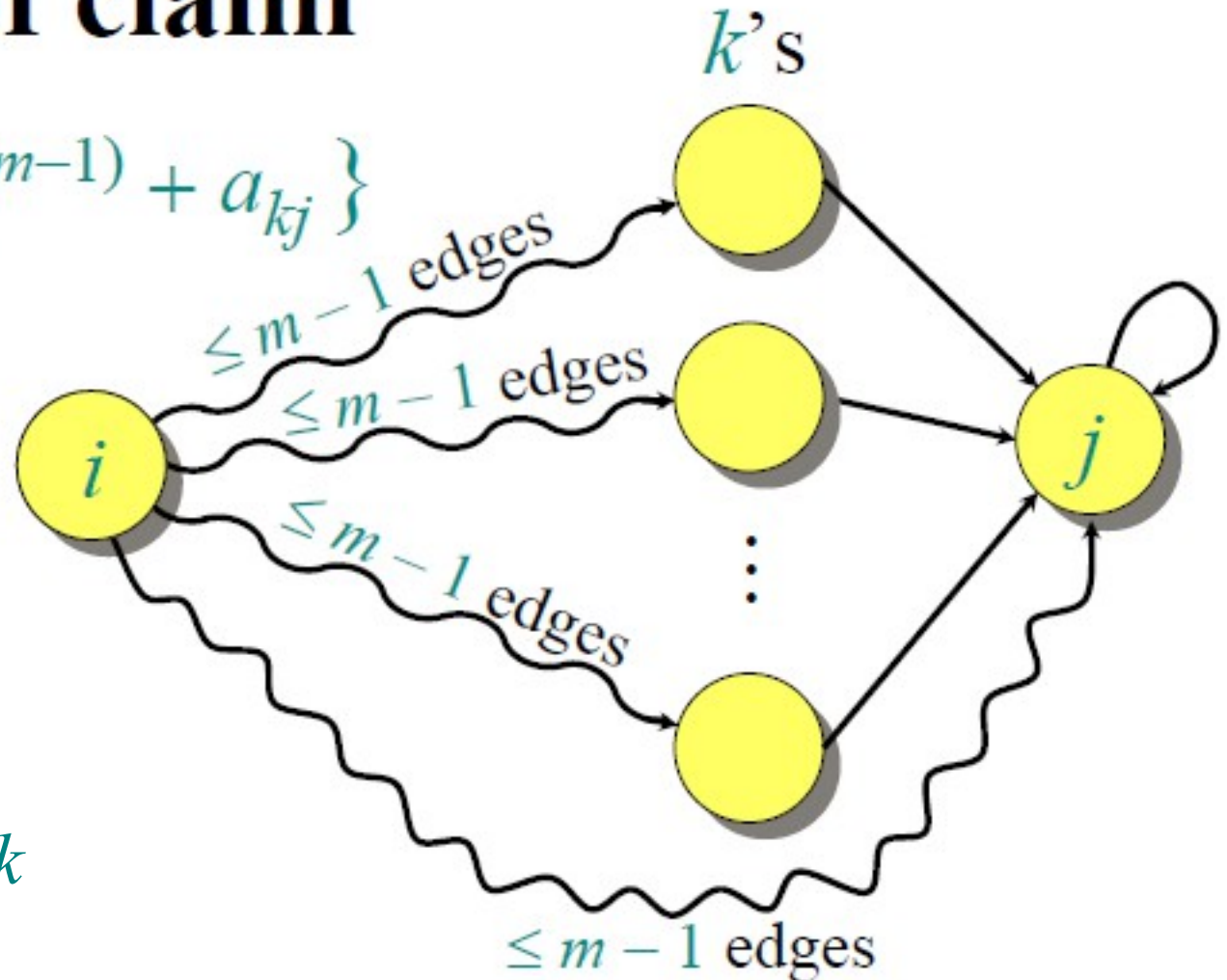


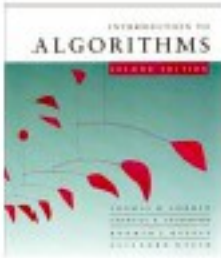
Proof of claim

$$d_{ij}^{(m)} = \min_k \{ d_{ik}^{(m-1)} + a_{kj} \}$$

We already proved the *optimal substructure* of the shortest paths problem

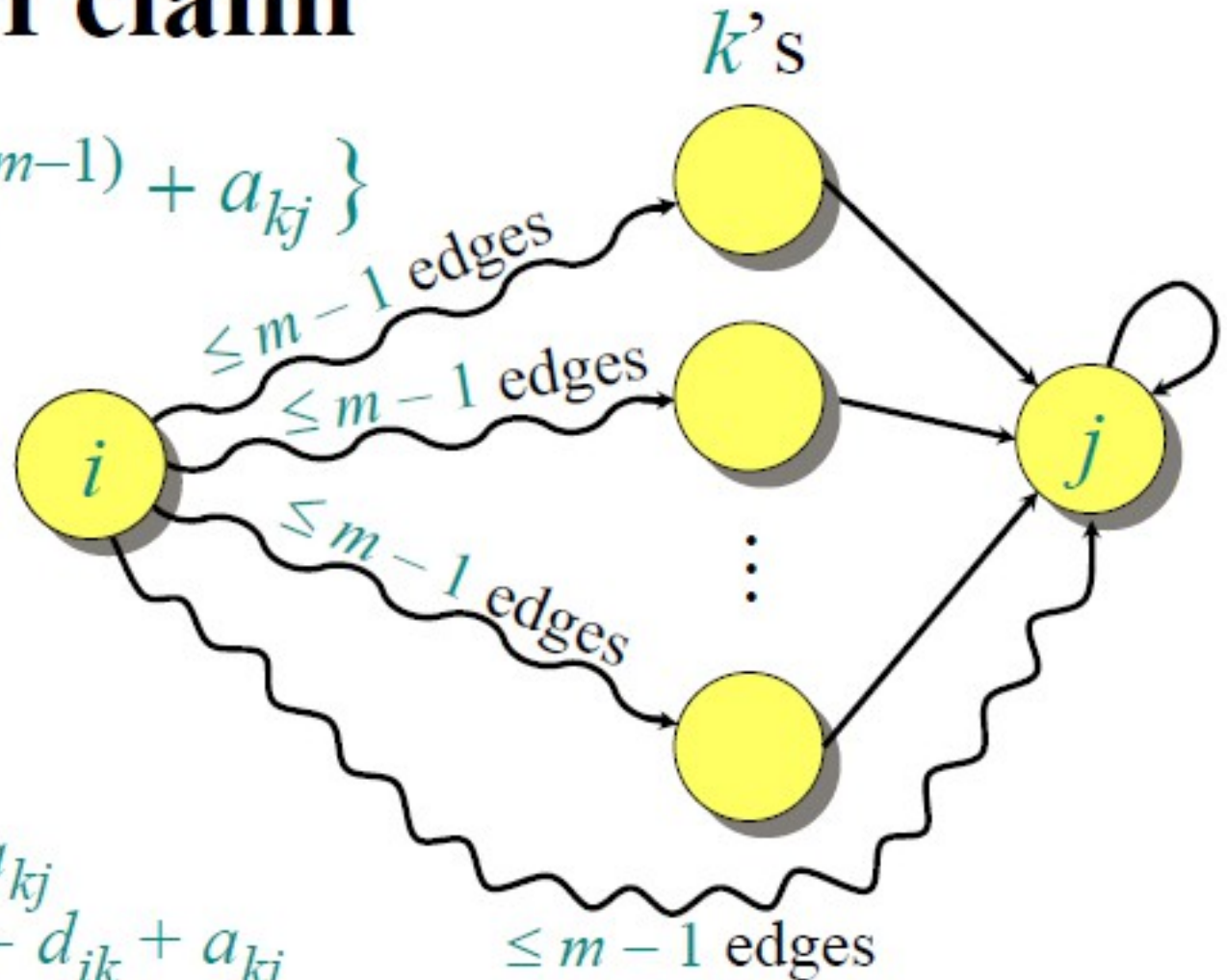
$d_{ik}^{(m-1)}$ is the minimum weight to get from i to k in $m-1$ steps





Proof of claim

$$d_{ij}^{(m)} = \min_k \{ d_{ik}^{(m-1)} + a_{kj} \}$$

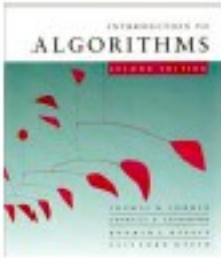


Relaxation!

for $k \leftarrow 1$ to n

do if $d_{ij} > d_{ik} + a_{kj}$
then $d_{ij} \leftarrow d_{ik} + a_{kj}$

Looking at all paths with $m-1$ edges, we go forward to find the shortest one with m edges



Proof of claim

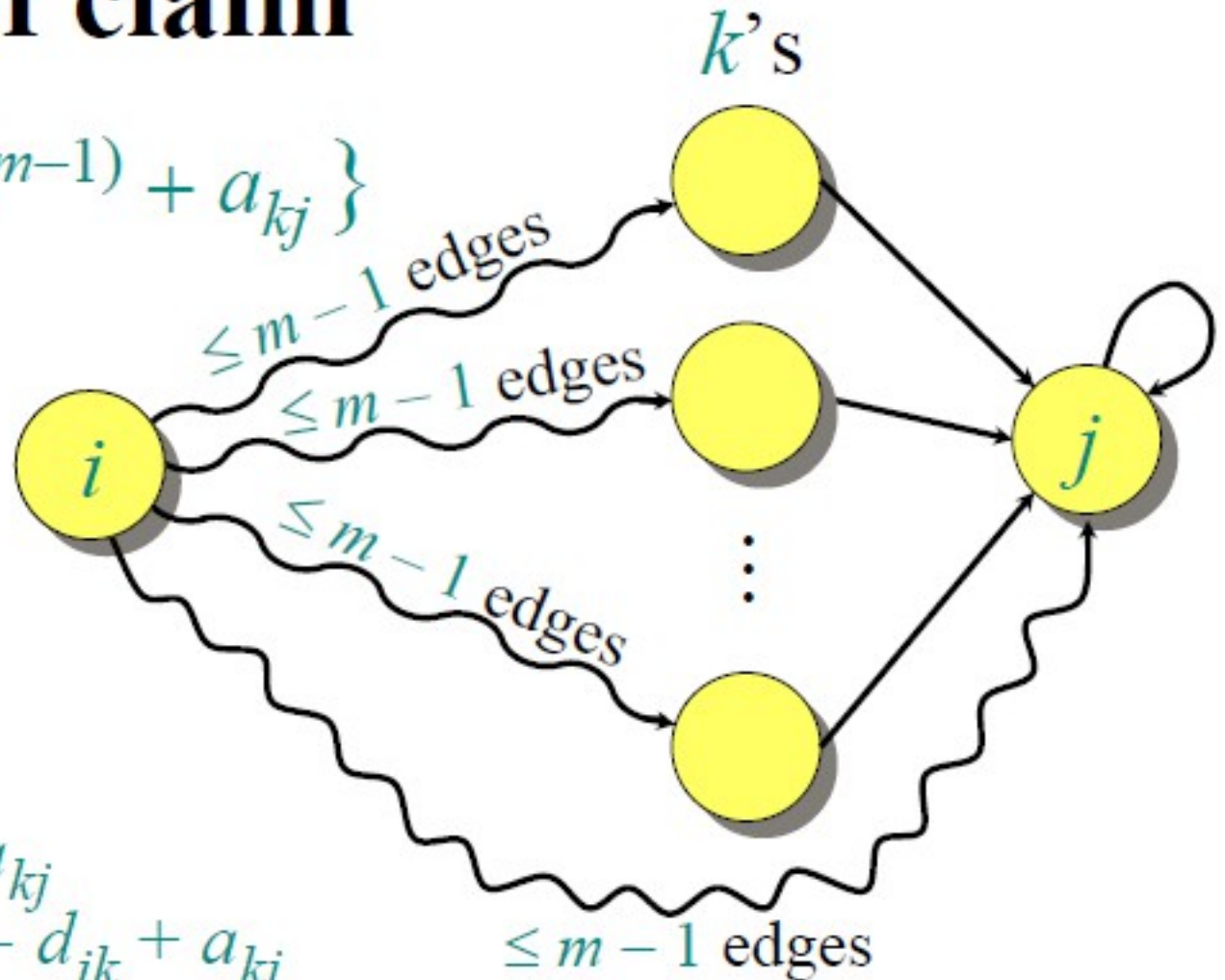
$$d_{ij}^{(m)} = \min_k \{ d_{ik}^{(m-1)} + a_{kj} \}$$



Relaxation!

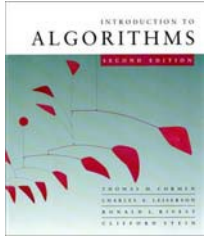
for $k \leftarrow 1$ to n

do if $d_{ij} > d_{ik} + a_{kj}$
 then $d_{ij} \leftarrow d_{ik} + a_{kj}$



Note: No negative-weight cycles implies

$$\delta(i, j) = d_{ij}^{(n-1)} = d_{ij}^{(n)} = d_{ij}^{(n+1)} = \dots$$

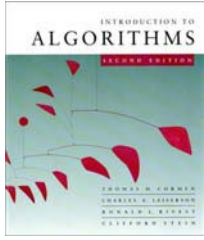


Matrix multiplication

Compute $C = A \cdot B$, where C , A , and B are $n \times n$ matrices:

$$c_{ij} = \sum_{k=1}^n a_{ik} b_{kj} .$$

Time = $\Theta(n^3)$ using the standard algorithm.



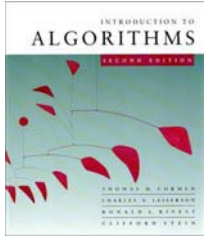
Matrix multiplication

Compute $C = A \cdot B$, where C , A , and B are $n \times n$ matrices:

$$c_{ij} = \sum_{k=1}^n a_{ik} b_{kj}.$$

Time = $\Theta(n^3)$ using the standard algorithm.

What if we map “+” \rightarrow “min” and “.” \rightarrow “+”?



Matrix multiplication

Compute $C = A \cdot B$, where C , A , and B are $n \times n$ matrices:

$$c_{ij} = \sum_{k=1}^n a_{ik} b_{kj}.$$

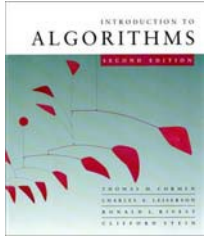
Time = $\Theta(n^3)$ using the standard algorithm.

What if we map “+” \rightarrow “min” and “.” \rightarrow “+”?

$$c_{ij} = \min_k \{a_{ik} + b_{kj}\}.$$

Thus, $D^{(m)} = D^{(m-1)} \times A$.

$$\text{Identity matrix} = I = \begin{pmatrix} 0 & \infty & \infty & \infty \\ \infty & 0 & \infty & \infty \\ \infty & \infty & 0 & \infty \\ \infty & \infty & \infty & 0 \end{pmatrix} = D^0 = (d_{ij}^{(0)}).$$



Matrix multiplication (continued)

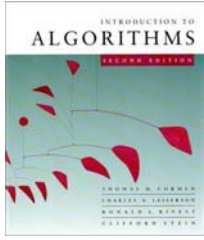
The $(\min, +)$ multiplication is *associative*, and with the real numbers, it forms an algebraic structure called a *closed semiring*.

Consequently, we can compute

$$\begin{aligned} D^{(1)} &= D^{(0)} \cdot A = A^1 \\ D^{(2)} &= D^{(1)} \cdot A = A^2 \\ &\vdots \\ D^{(n-1)} &= D^{(n-2)} \cdot A = A^{n-1}, \end{aligned}$$

yielding $D^{(n-1)} = (\delta(i, j))$.

Time = $\Theta(n \cdot n^3) = \Theta(n^4)$. No better than $n \times$ B-F.



Improved matrix multiplication algorithm

Repeated squaring: $A^{2^k} = A^k \times A^k$.

Compute $A^2, A^4, \dots, A^{2^{\lceil \lg(n-1) \rceil}}$.

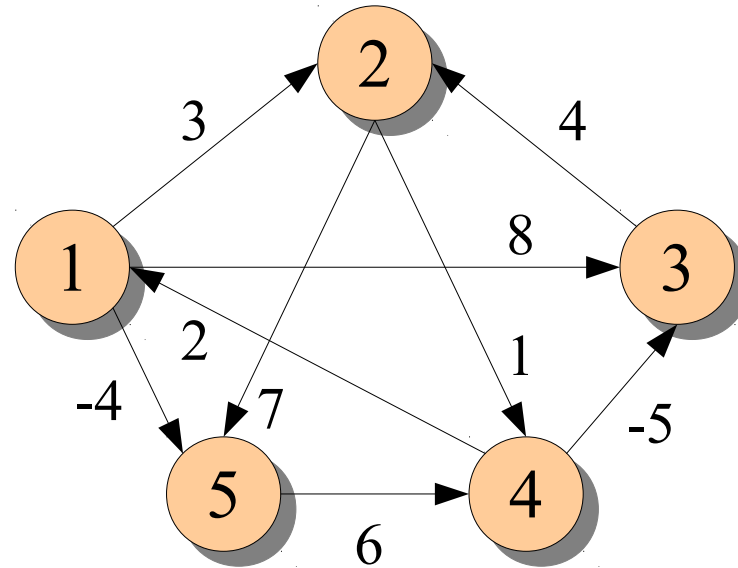
$O(\lg n)$ squarings

Note: $A^{n-1} = A^n = A^{n+1} = \dots$.

Time = $\Theta(n^3 \lg n)$.

To detect negative-weight cycles, check the diagonal for negative values in $O(n)$ additional time.

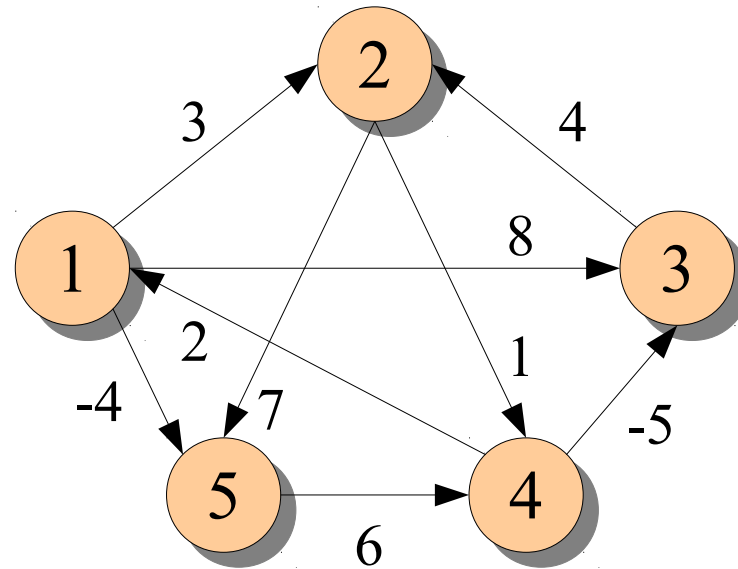
Matrix Multiplication Algorithm



$$d_{ij} = \begin{cases} a_{ij} & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases}$$

$$D^{(1)} = A = \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$

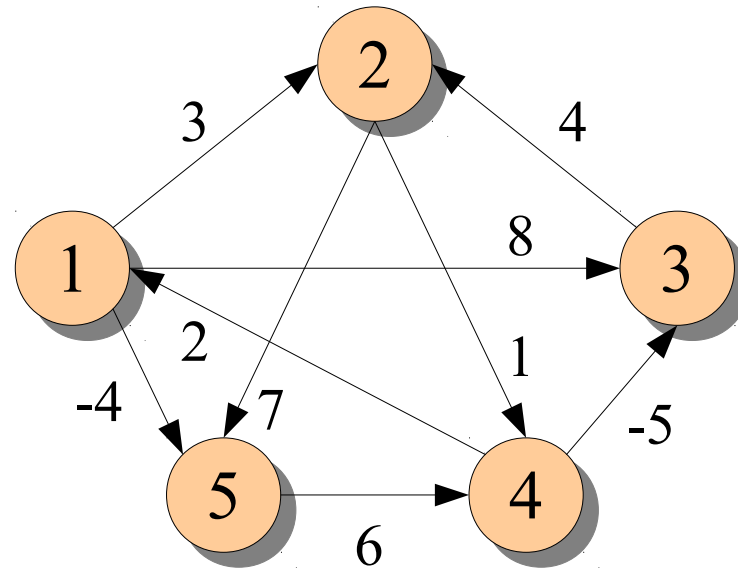
Matrix Multiplication Algorithm



$$d_{ij} = \min_k (d_{ik} + d_{kj}) \quad \min(0 + \infty, 3 + 1, 8 + \infty, \infty + 0, -4 + 6)$$

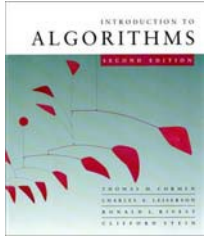
$$D^{(2)} = \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 3 & 8 & 2 & -4 \\ 3 & 0 & -4 & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & \infty & 1 & 6 & 0 \end{bmatrix}$$

Matrix Multiplication Algorithm



$$d_{ij} = \min_k (d_{ik} + d_{kj})$$

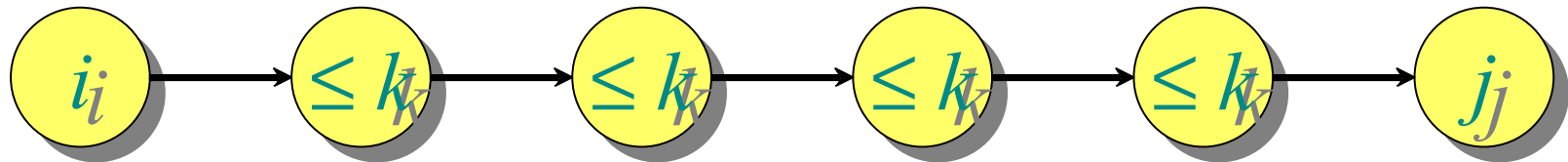
$$D^{(4)} = \begin{bmatrix} 0 & 3 & 8 & 2 & -4 \\ 3 & 0 & -4 & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & \infty & 1 & 6 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 1 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{bmatrix}$$



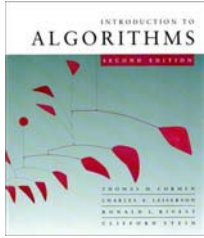
Floyd-Warshall algorithm

Also dynamic programming, but faster!

Define $c_{ij}^{(k)}$ = weight of a shortest path from i to j with intermediate vertices belonging to the set $\{1, 2, \dots, k\}$.

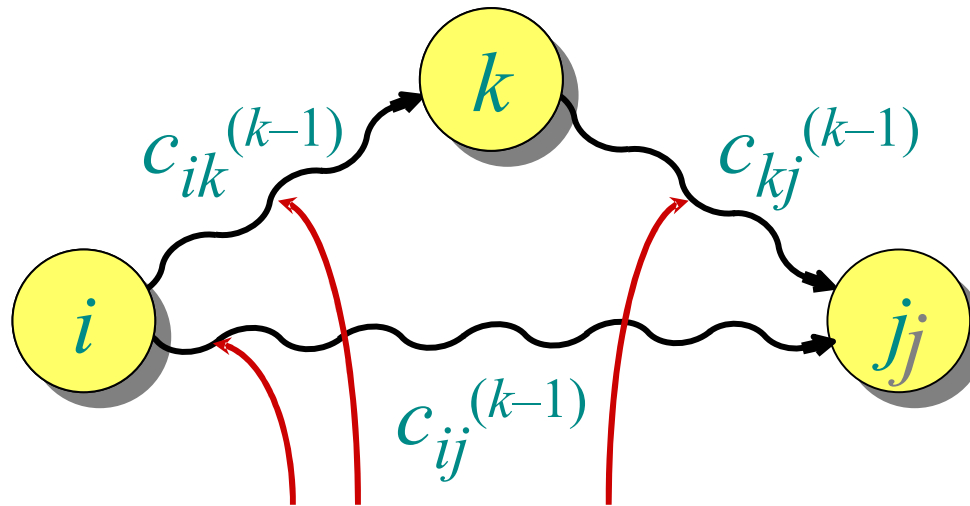


Thus, $\delta(i, j) = c_{ij}^{(n)}$. Also, $c_{ij}^{(0)} = a_{ij}$.

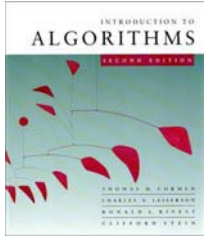


Floyd-Warshall recurrence

$$c_{ij}^{(k)} = \min \{c_{ij}^{(k-1)}, c_{ik}^{(k-1)} + c_{kj}^{(k-1)}\}$$

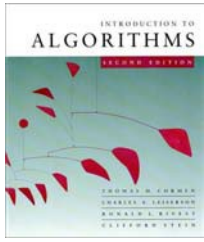


intermediate vertices in $\{1, 2, \dots, k\}$



Pseudocode for Floyd-Warshall

```
for  $k \leftarrow 1$  to  $n$ 
  do for  $i \leftarrow 1$  to  $n$ 
    do for  $j \leftarrow 1$  to  $n$ 
      do if  $c_{ij} > c_{ik} + c_{kj}$ 
        then  $c_{ij} \leftarrow c_{ik} + c_{kj}$  } relaxation
```



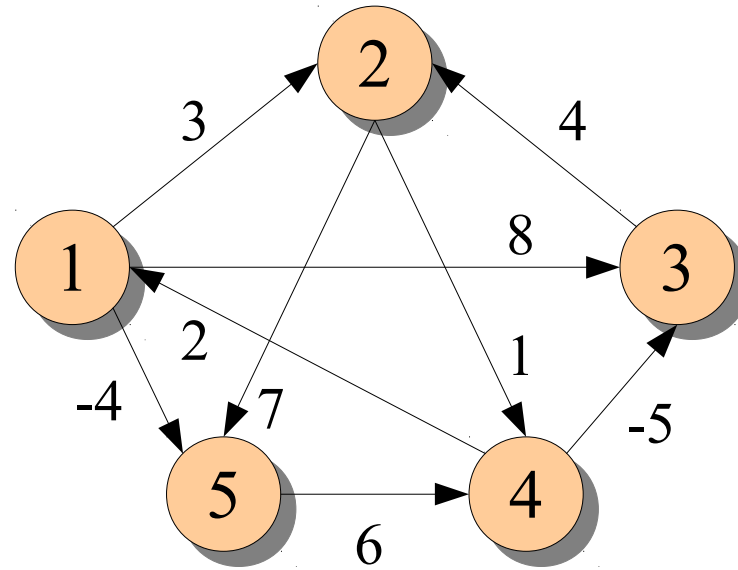
Pseudocode for Floyd-Warshall

```
for  $k \leftarrow 1$  to  $n$ 
  do for  $i \leftarrow 1$  to  $n$ 
    do for  $j \leftarrow 1$  to  $n$ 
      do if  $c_{ij} > c_{ik} + c_{kj}$ 
        then  $c_{ij} \leftarrow c_{ik} + c_{kj}$  } relaxation
```

Notes:

- Okay to omit superscripts, since extra relaxations can't hurt.
- Runs in $\Theta(n^3)$ time.
- Simple to code.
- Efficient in practice.

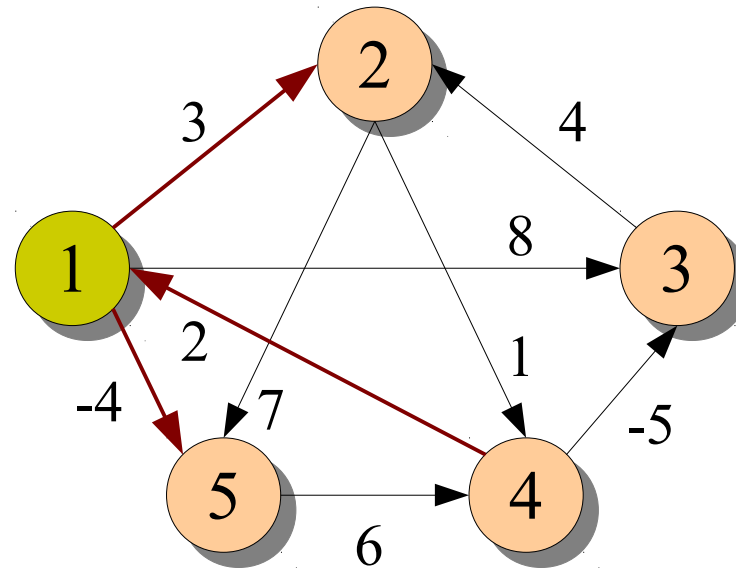
Floyd-Warshall Algorithm



$$c_{ij}^{(0)} = \begin{cases} a_{ij} & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases}$$

$$C^{(0)} = \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$

Floyd-Warshall Algorithm

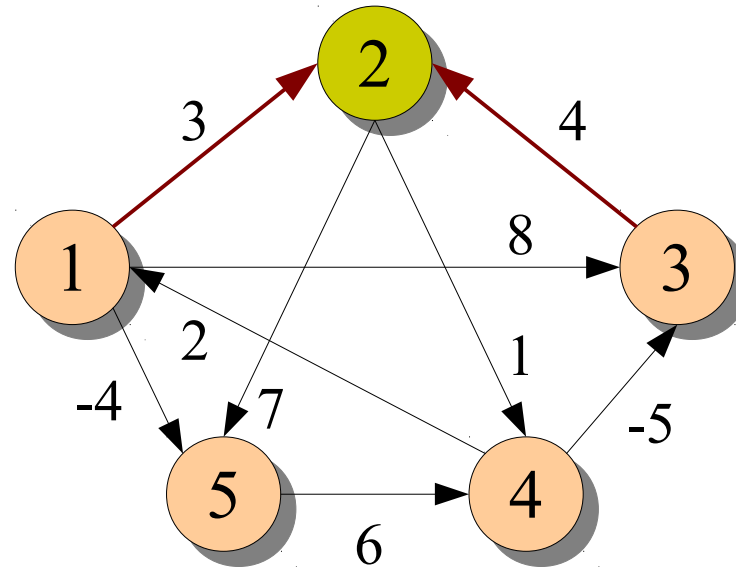


$$c_{ij}^{(k)} = \min \left(c_{ij}^{(k-1)}, c_{ik}^{(k-1)} + c_{kj}^{(k-1)} \right) \quad c_{42}^{(1)} = \min \left(c_{42}^{(0)}, c_{41}^{(0)} + c_{12}^{(0)} \right) = \min \left(\infty, 2+3 \right)$$

$$C^{(0)} = \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix} \quad C^{(1)} = \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$

$$c_{45}^{(1)} = \min \left(c_{45}^{(0)}, c_{41}^{(0)} + c_{15}^{(0)} \right) = \min \left(\infty, 2-4 \right)$$

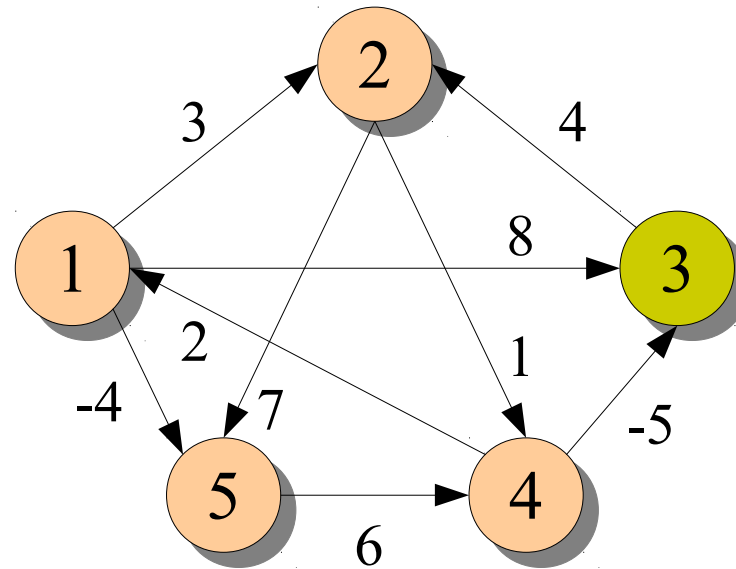
Floyd-Warshall Algorithm



$$c_{ij}^{(k)} = \min \left(c_{ij}^{(k-1)}, c_{ik}^{(k-1)} + c_{kj}^{(k-1)} \right) \quad c_{14}^{(2)} = \min \left(c_{14}^{(1)}, c_{12}^{(1)} + c_{24}^{(1)} \right) = \min \left(\infty, 3 + 1 \right)$$

$$C^{(1)} = \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix} \quad C^{(2)} = \begin{bmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$

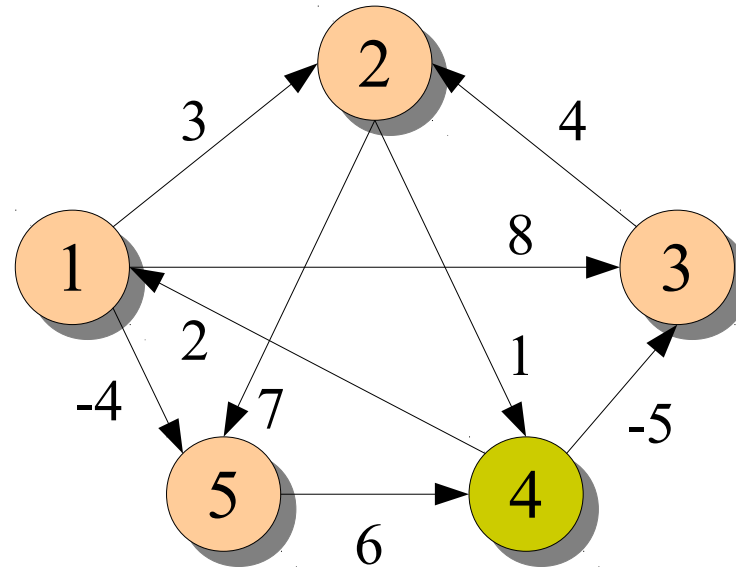
Floyd-Warshall Algorithm



$$c_{ij}^{(k)} = \min \left(c_{ij}^{(k-1)}, c_{ik}^{(k-1)} + c_{kj}^{(k-1)} \right)$$

$$C^{(2)} = \begin{bmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix} \quad C^{(3)} = \begin{bmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$

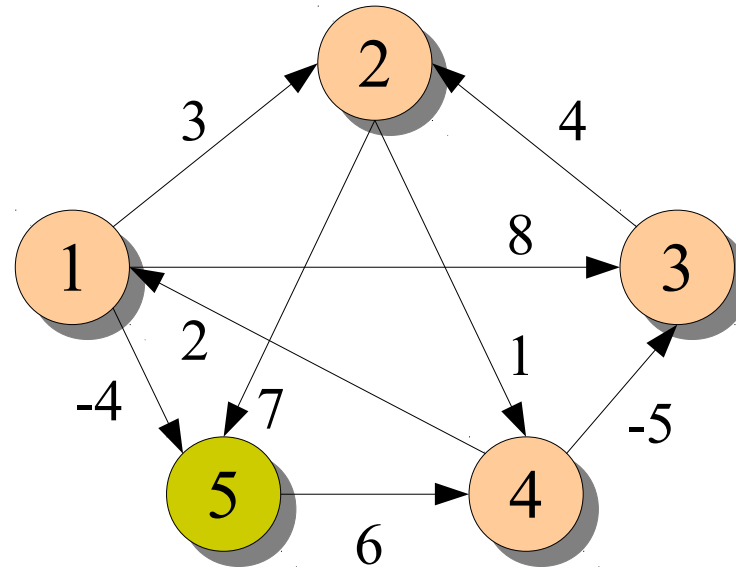
Floyd-Warshall Algorithm



$$c_{ij}^{(k)} = \min \left(c_{ij}^{(k-1)}, c_{ik}^{(k-1)} + c_{kj}^{(k-1)} \right)$$

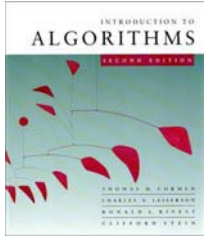
$$C^{(3)} = \begin{bmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix} \quad C^{(4)} = \begin{bmatrix} 0 & 3 & -1 & 4 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{bmatrix}$$

Floyd-Warshall Algorithm



$$c_{ij}^{(k)} = \min \left(c_{ij}^{(k-1)}, c_{ik}^{(k-1)} + c_{kj}^{(k-1)} \right)$$

$$C^{(4)} = \begin{bmatrix} 0 & 3 & -1 & 4 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{bmatrix} \quad C^{(5)} = \begin{bmatrix} 0 & 1 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{bmatrix}$$



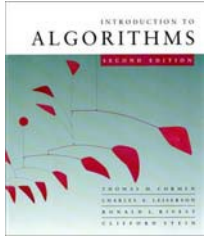
Transitive closure of a directed graph

Compute $t_{ij} = \begin{cases} 1 & \text{if there exists a path from } i \text{ to } j, \\ 0 & \text{otherwise.} \end{cases}$

IDEA: Use Floyd-Warshall, but with (\vee, \wedge) instead of $(\min, +)$:

$$t_{ij}^{(k)} = t_{ij}^{(k-1)} \vee (t_{ik}^{(k-1)} \wedge t_{kj}^{(k-1)}).$$

Can we reach from $(i \rightarrow k \text{ and } k \rightarrow j)$ or $i \rightarrow j$



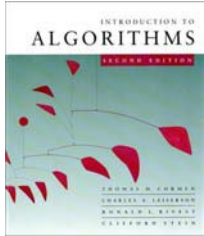
Transitive closure of a directed graph

Compute $t_{ij} = \begin{cases} 1 & \text{if there exists a path from } i \text{ to } j, \\ 0 & \text{otherwise.} \end{cases}$

IDEA: Use Floyd-Warshall, but with (\vee, \wedge) instead of $(\min, +)$:

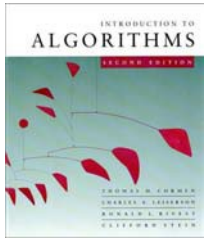
$$t_{ij}^{(k)} = t_{ij}^{(k-1)} \vee (t_{ik}^{(k-1)} \wedge t_{kj}^{(k-1)}).$$

Time = $\Theta(n^3)$.



Graph reweighting

Theorem. Given a function $h : V \rightarrow \mathbb{R}$, *reweight* each edge $(u, v) \in E$ by $w_h(u, v) = w(u, v) + h(u) - h(v)$. Then, for any two vertices, all paths between them are reweighted by the same amount.

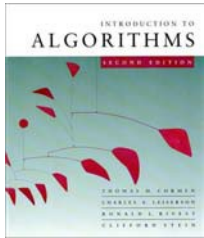


Graph reweighting

Theorem. Given a function $h : V \rightarrow \mathbb{R}$, *reweight* each edge $(u, v) \in E$ by $w_h(u, v) = w(u, v) + h(u) - h(v)$. Then, for any two vertices, all paths between them are reweighted by the same amount.

Proof. Let $p = v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_k$ be a path in G . We have

$$\begin{aligned} w_h(p) &= \sum_{i=1}^{k-1} w_h(v_i, v_{i+1}) \\ &= \sum_{i=1}^{k-1} (w(v_i, v_{i+1}) + h(v_i) - h(v_{i+1})) \end{aligned}$$



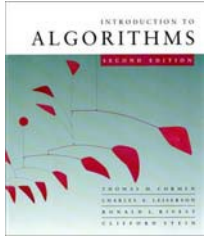
Graph reweighting

Theorem. Given a function $h : V \rightarrow \mathbb{R}$, *reweight* each edge $(u, v) \in E$ by $w_h(u, v) = w(u, v) + h(u) - h(v)$. Then, for any two vertices, all paths between them are reweighted by the same amount.

Proof. Let $p = v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_k$ be a path in G . We have

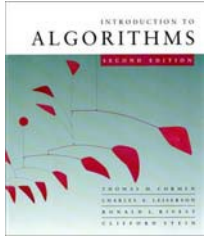
$$\begin{aligned} w_h(p) &= \sum_{i=1}^{k-1} w_h(v_i, v_{i+1}) \\ &= \sum_{i=1}^{k-1} (w(v_i, v_{i+1}) + h(v_i) - h(v_{i+1})) \\ &= \sum_{i=1}^{k-1} w(v_i, v_{i+1}) + h(v_1) - h(v_k) \\ &= w(p) + h(v_1) - h(v_k). \end{aligned}$$

Same amount!



Shortest paths in reweighted graphs

Corollary. $\delta_h(u, v) = \delta(u, v) + h(u) - h(v)$. □

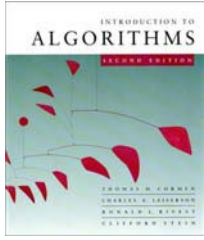


Shortest paths in reweighted graphs

Corollary. $\delta_h(u, v) = \delta(u, v) + h(u) - h(v)$. □

IDEA: Find a function $h : V \rightarrow \mathbb{R}$ such that $w_h(u, v) \geq 0$ for all $(u, v) \in E$. Then, run Dijkstra's algorithm from each vertex on the reweighted graph.

NOTE: $w_h(u, v) \geq 0$ iff $h(v) - h(u) \leq w(u, v)$.



Johnson's algorithm

1. Find a function $h : V \rightarrow \mathbb{R}$ such that $w_h(u, v) \geq 0$ for all $(u, v) \in E$ by using Bellman-Ford to solve the difference constraints $h(v) - h(u) \leq w(u, v)$, or determine that a negative-weight cycle exists.
 - Time = $O(VE)$.
2. Run Dijkstra's algorithm using w_h from each vertex $u \in V$ to compute $\delta_h(u, v)$ for all $v \in V$.
 - Time = $O(VE \lg V)$
3. For each $(u, v) \in V \times V$, compute
$$\delta(u, v) = \delta_h(u, v) - h(u) + h(v) .$$
 - Time = $O(V^2)$.

Total time = $O(VE \lg V)$

Recap

- Matrix Multiplication Algorithm
- Floyd-Warshall Algorithm
- Johnson's Algorithm