CMP448: Algorithms

Lecture 16: Network Flow

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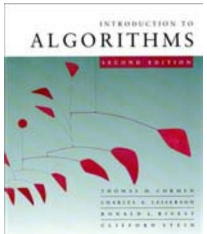
Spring 2013

Agenda

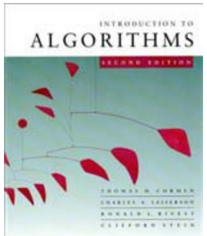
- Flow Networks
- Maximum Flow Problem
- Flow Notation
- Properties of Flow
- Cuts
- Residual Networks
- Augmented Paths

Acknowledgment
A lot of slides adapted from the slides of Charles Leiserson.



Flow networks

Definition. A *flow network* is a directed graph $G = (V, E)$ with two distinguished vertices: a *source* s and a *sink* t. Each edge $(u, v) \in E$ has a nonnegative *capacity* $c(u, v)$. If $(u, v) \notin E$, then $c(u, v) = 0$.



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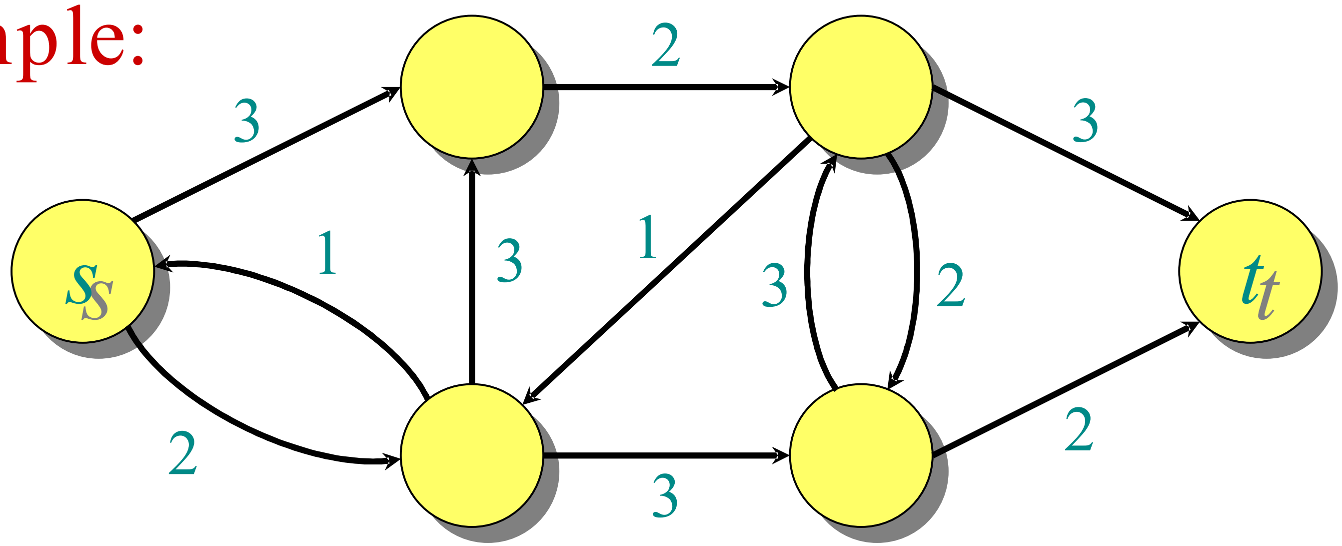
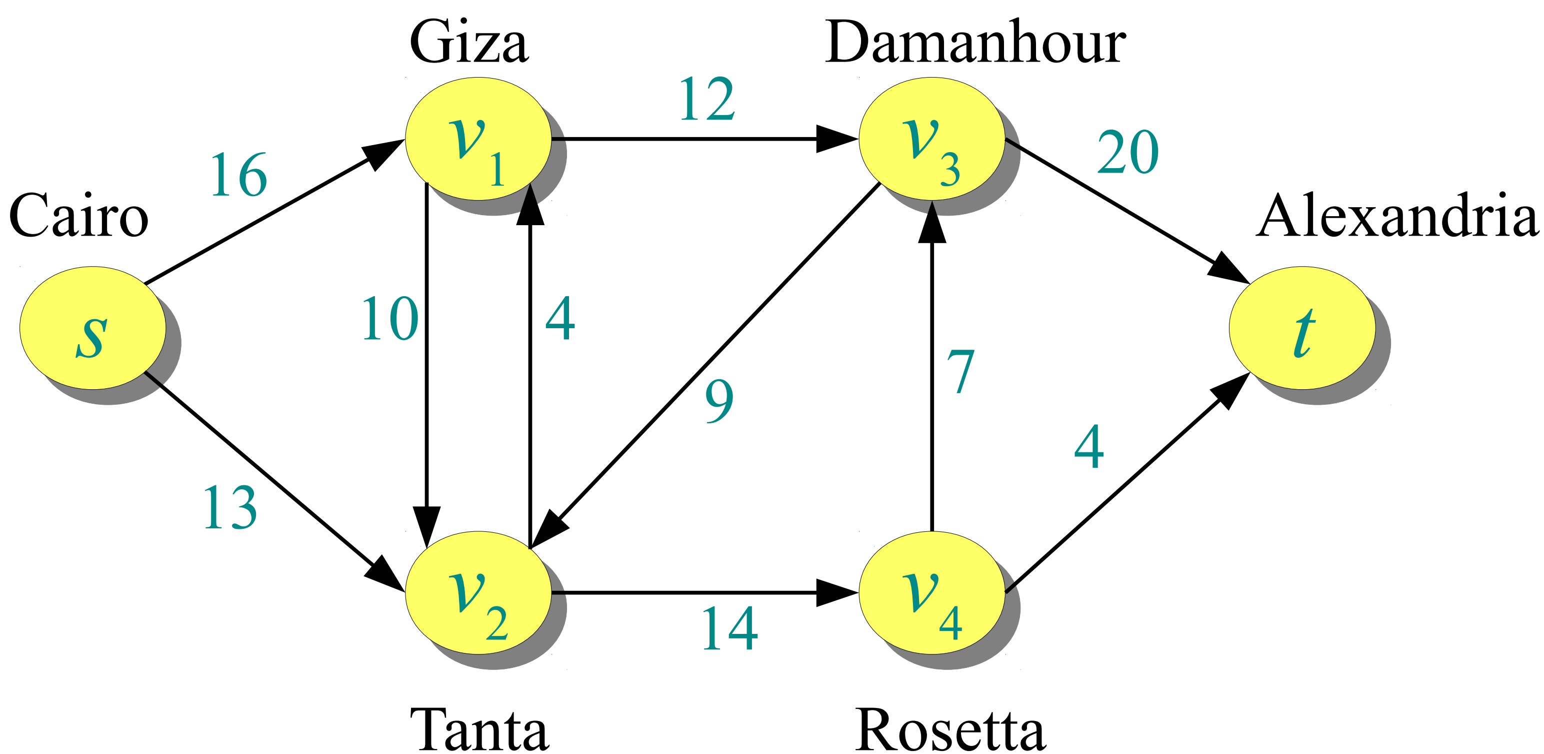
Example:

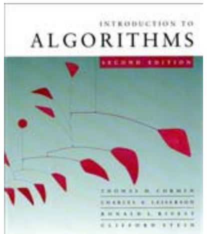


Flow Networks

Tasty Bread Co.



- Bread produced at *Cairo* and consumed at *Alexandria*.
- No storage at intermediate cities.
- Each link has maximum *capacity* of bread that can be shipped.
- What is the *maximum* amount to be produced?



Flow networks

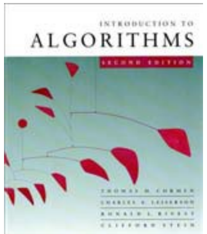
Definition. A *positive flow* on G is a function $p : V \times V \to \mathbb{R}$ satisfying the following:

- *Capacity constraint:* For all $u, v \in V$,

$$0 \le p(u, v) \le c(u, v).$$

- *Flow conservation:* For all $u \in V - \{s, t\}$,

$$\sum_{v \in V} p(u,v) - \sum_{v \in V} p(v,u) = 0.$$



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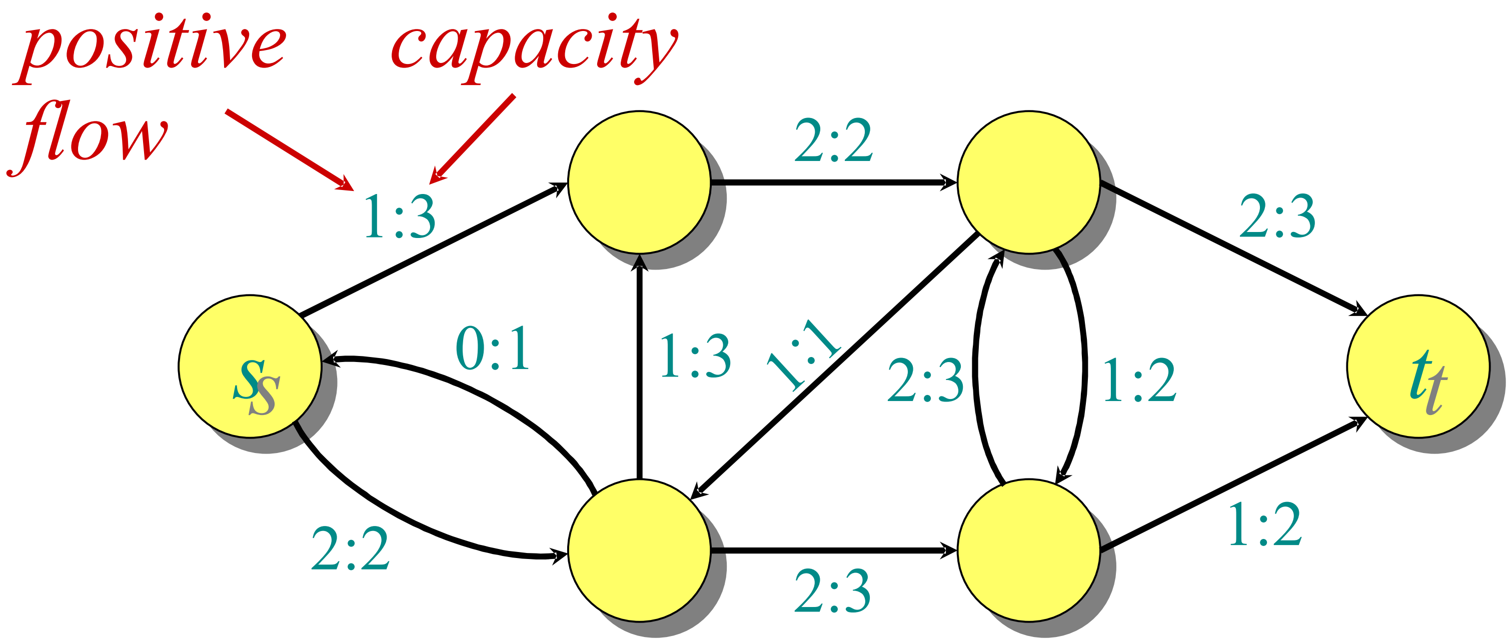
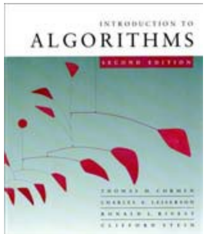
The *value* of a flow is the net flow out of the source:
$$\sum_{v \in V} p(s,v) - \sum_{v \in V} p(v,s) .$$

A flow on a network





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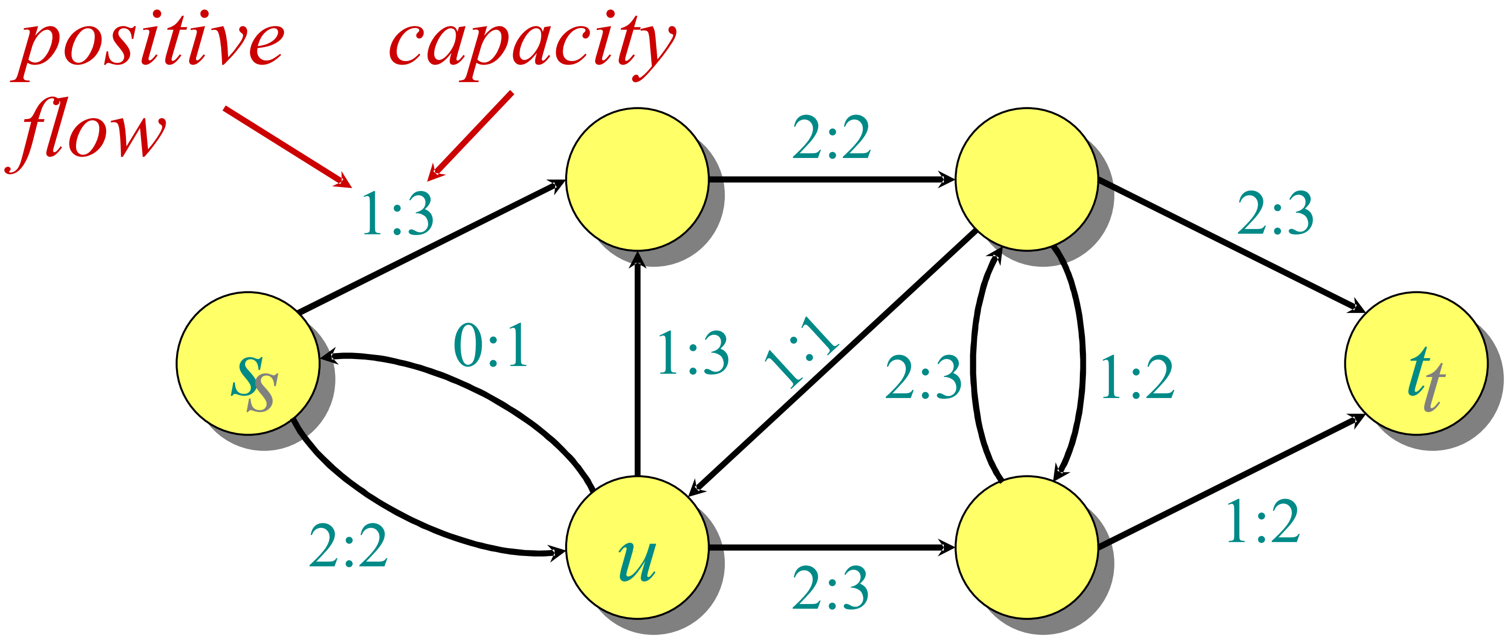
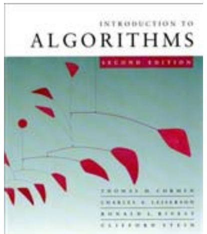


Flow conservation (like Kirchoff's current law):



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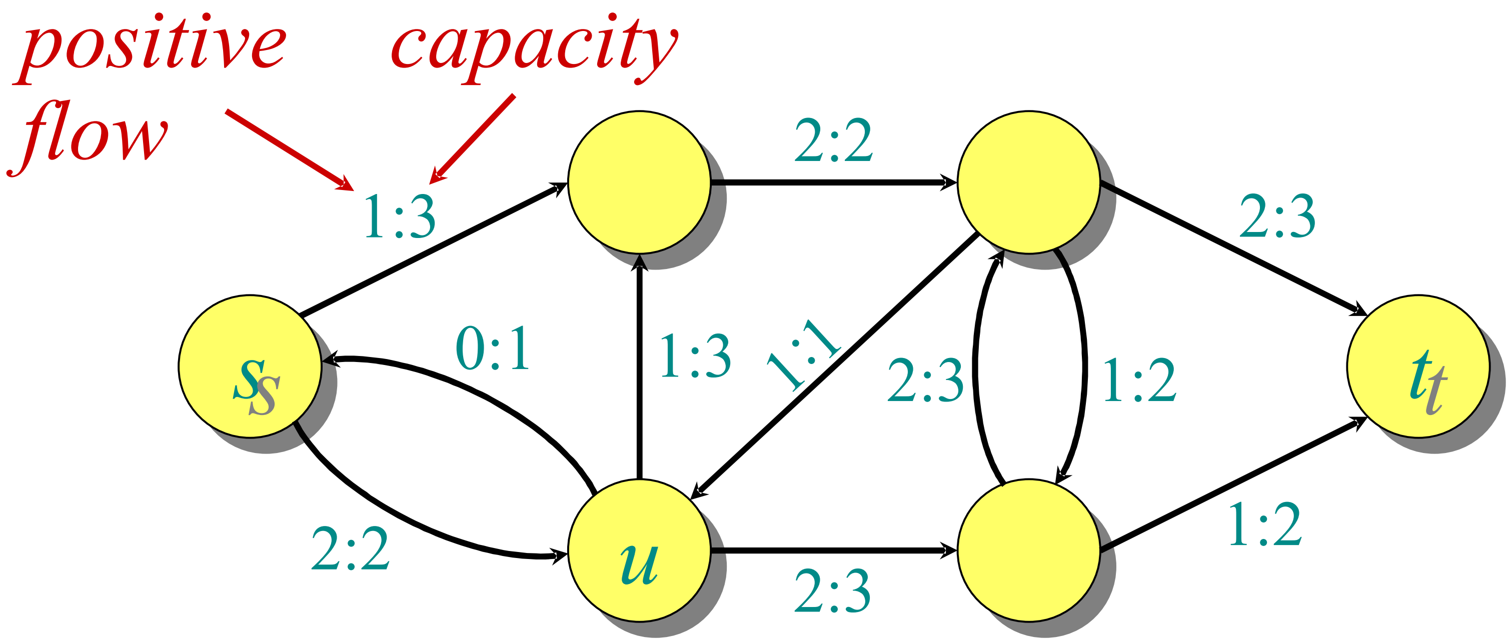
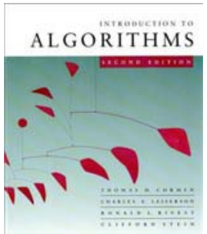


Flow conservation (like Kirchoff's current law):

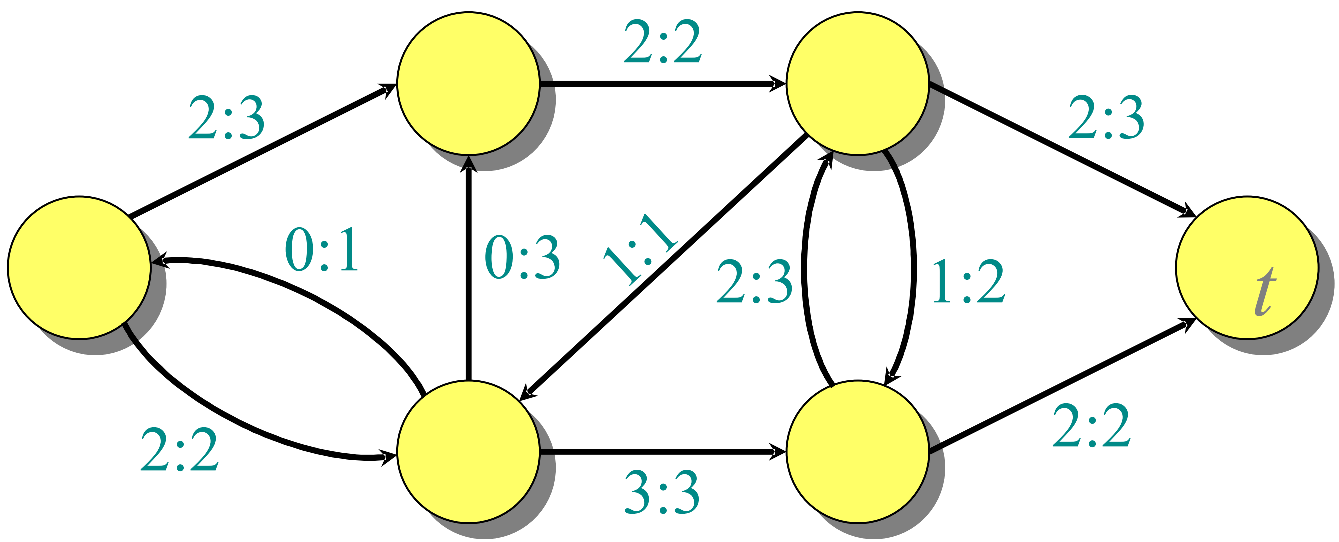
- Flow into u is $2 + 1 = 3$.
- Flow out of u is $0 + 1 + 2 = 3$.

The value of this flow is $1 - 0 + 2 = 3$.



The maximum-flow problem

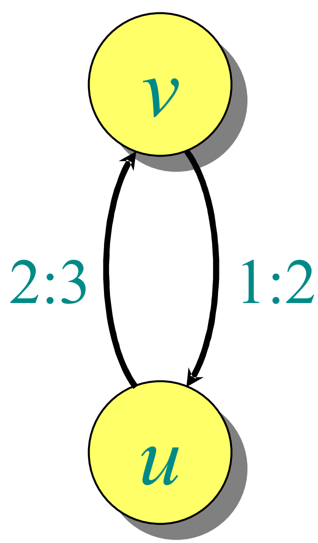
Maximum-flow problem: Given a flow network G, find a flow of maximum value on G.

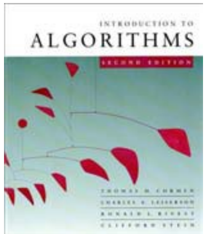


The value of the maximum flow is 4.

Flow cancellation

Without loss of generality, positive flow goes either from *u* to *v*, or from *v* to *u*, but not both.





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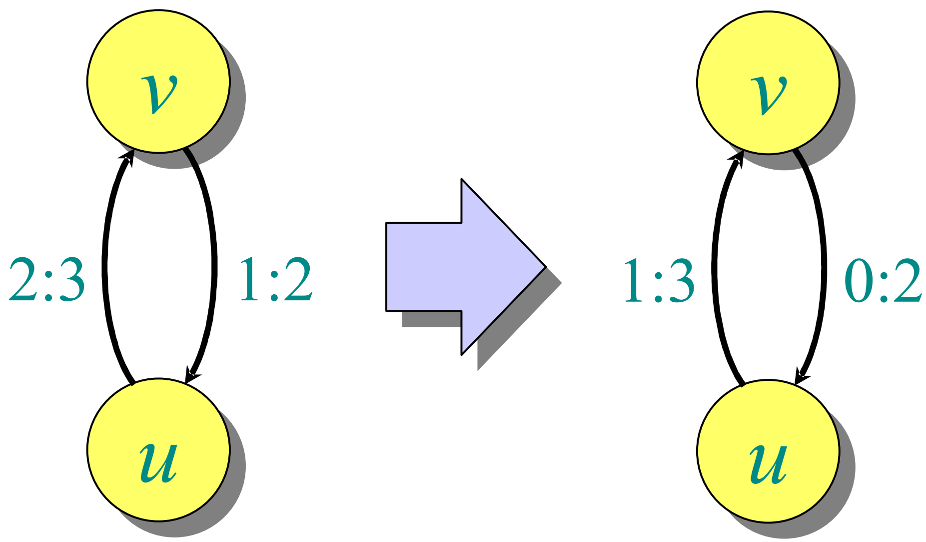
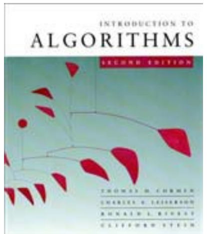


Net flow from u to v in both cases is 1.

The capacity constraint and flow conservation are preserved by this transformation.

INTUITION: View flow as a *rate*, not a *quantity*.



A notational simplification

IDEA: Work with the net flow between two vertices, rather than with the positive flow.

Definition. A *(net) flow* on G is a function $f : V \times V \to \mathbb{R}$ satisfying the following:

- *Capacity constraint:* For all $u, v \in V$,

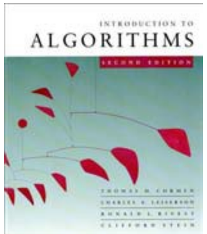
$$f(u, v) \le c(u, v).$$

- *Flow conservation:* For all $u \in V - \{s, t\}$,

$$\sum_{v \in V} f(u, v) = 0.$$

- *Skew symmetry:* For all $u, v \in V$,

$$f(u, v) = -f(v, u).$$



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$$\sum_{v \in V} f(u, v) = 0.$$
← *One summation instead of two.*
- *Skew symmetry:* For all $u, v \in V$,
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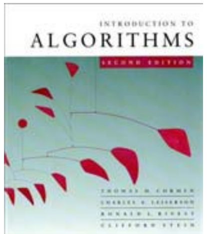
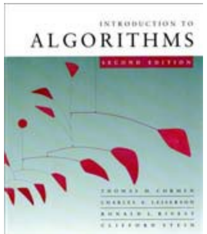


Equivalence of definitions

Theorem. The two definitions are equivalent.



Equivalence of definitions

Theorem. The two definitions are equivalent.

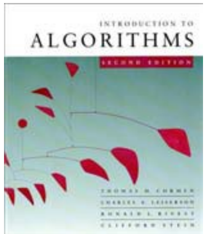
Proof. ($\Rightarrow$) Let $f(u, v) = p(u, v) - p(v, u)$.

- *Capacity constraint:* Since $p(u, v) \le c(u, v)$ and $p(v, u) \ge 0$, we have $f(u, v) \le c(u, v)$.
- *Flow conservation:*

$$\sum_{v \in V} f(u,v) = \sum_{v \in V} \big(p(u,v) - p(v,u)\big)$$
$$= \sum_{v \in V} p(u,v) - \sum_{v \in V} p(v,u) = 0$$

- *Skew symmetry:*

$$\begin{aligned} f(u, v) &= p(u, v) - p(v, u) \\ &= -\,(p(v, u) - p(u, v)) \\ &= -f(v, u). \end{aligned}$$



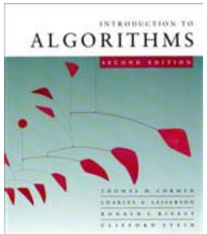
Proof (continued)

(⇐) Let

$$p(u, v) = \begin{cases} f(u, v) & \text{if } f(u, v) > 0, \\ 0 & \text{if } f(u, v) \le 0. \end{cases}$$

- *Capacity constraint:* By definition, $p(u, v) \ge 0$. Since $f(u, v) \le c(u, v)$, it follows that $p(u, v) \le c(u, v)$.
- *Flow conservation:* If $f(u, v) > 0$, then $p(u, v) - p(v, u) = f(u, v)$. If $f(u, v) \le 0$, then $p(u, v) - p(v, u) = -f(v, u) = f(u, v)$ by skew symmetry. Therefore,

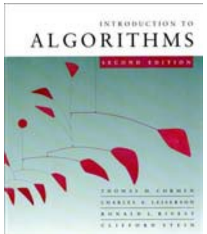
$$\sum_{v \in V} p(u,v) - \sum_{v \in V} p(v,u) = \sum_{v \in V} f(u,v) = 0 \ . \quad \square$$



Notation

Definition. The *value* of a flow f, denoted by $|f|$, is given by

$$|f| = \sum_{v \in V} f(s,v)$$
$$= f(s,V).$$



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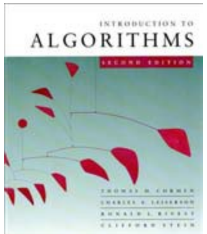
Implicit summation notation: A set used in an arithmetic formula represents a sum over the elements of the set.

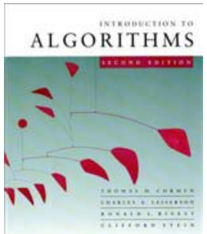
- Example — flow conservation:
  $f(u, V) = 0$ for all $u \in V - \{s, t\}$.



Simple properties of flow

Lemma.

- $f(X, X) = 0$,
- $f(X, Y) = -f(Y, X)$,
- $f(X \cup Y, Z) = f(X, Z) + f(Y, Z)$ if $X \cap Y = \emptyset$. $\square$



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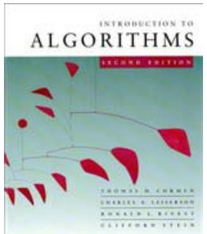
Theorem. $|f| = f(V, t)$.

The net flow from s equals the net flow into t



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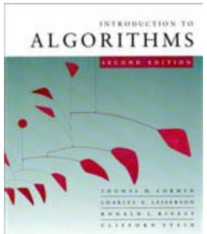
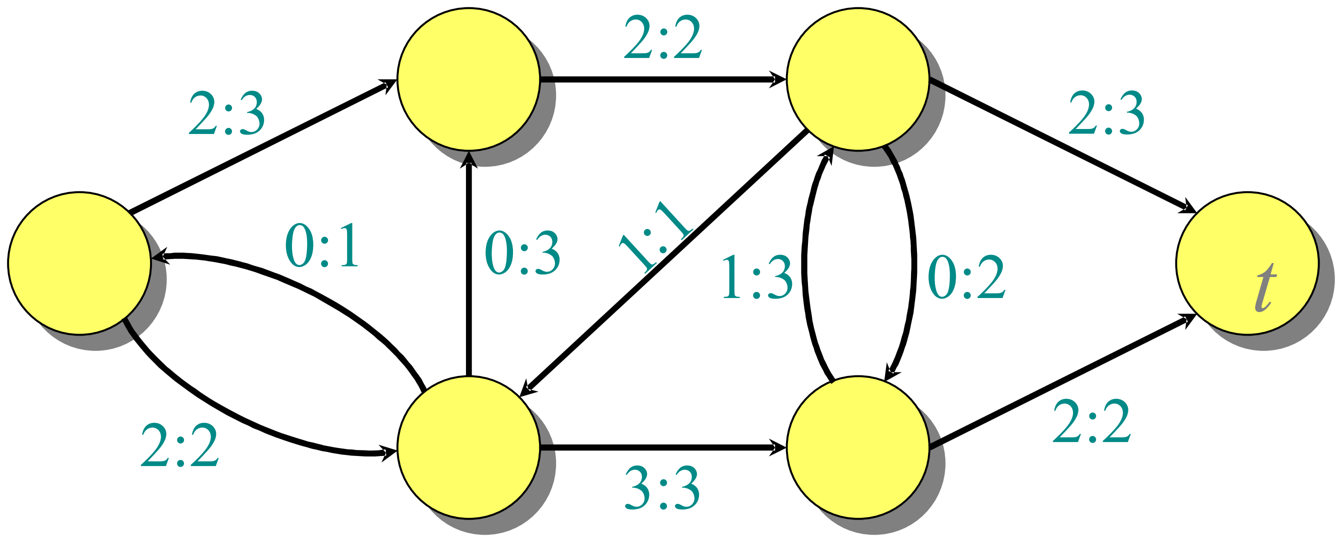
Theorem. $|f| = f(V, t)$. The net flow from s equals the net flow into t

Proof.

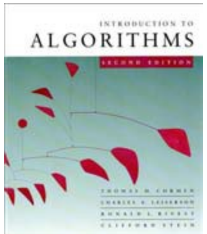
$$
\begin{aligned}
|f| &= f(s, V) \\
&= f(V, V) - f(V - s, V) && V = V - s \cup s \\
&= f(V, V - s) && f(V, V) = 0 \\
&= f(V, t) + f(V, V - s - t) && V - s = t \cup V - s - t \\
&= f(V, t). \quad \square && f(V, u) = 0 \text{ for } u \in V - \{s, t\}
\end{aligned}
$$



Flow into the sink

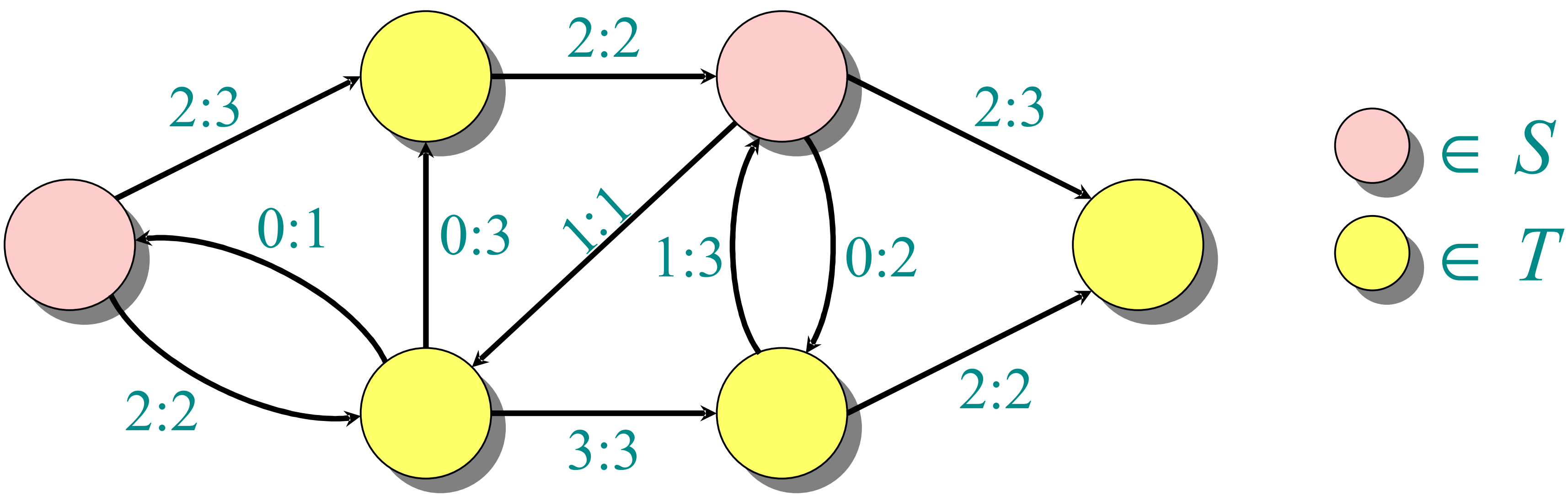


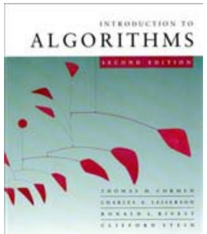
$|f| = f(s, V) = 4$ $\qquad$ $f(V, t) = 4$



Cuts

Definition. A *cut* (S, T) of a flow network $G = (V, E)$ is a partition of V such that $s \in S$ and $t \in T$. If f is a flow on G, then the *flow across the cut* is $f(S, T)$.





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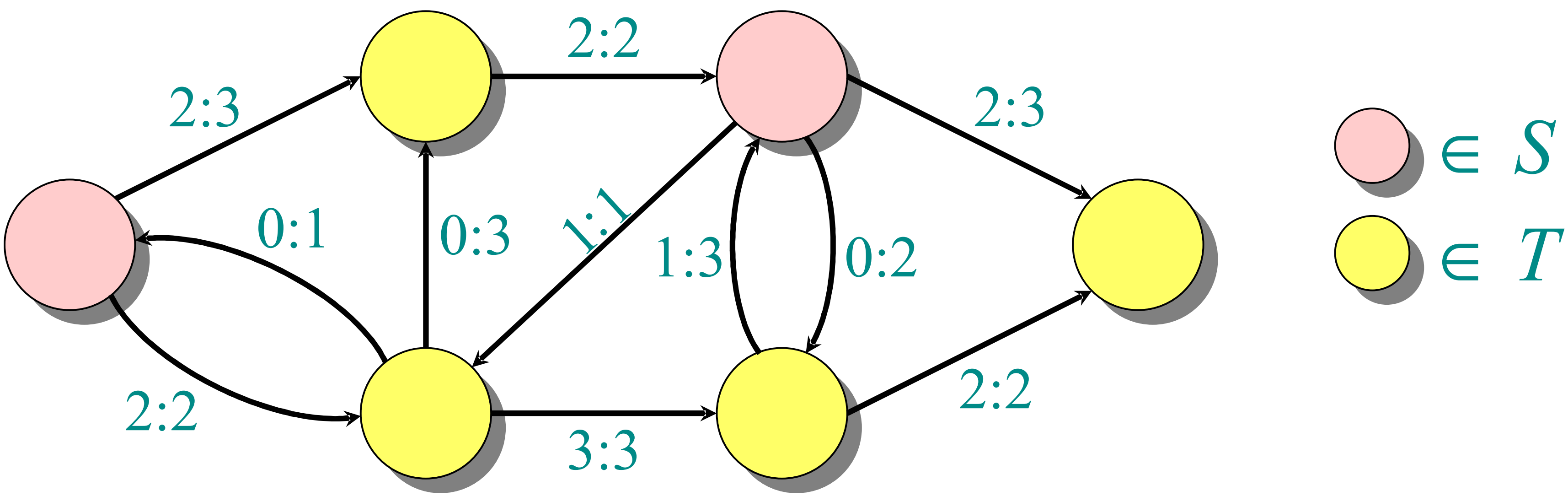
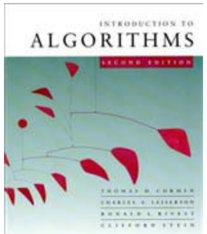


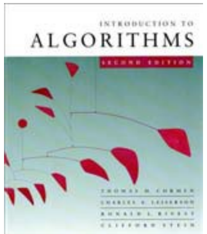
$$f(S, T) = (2 + 2) + (-2 + 1 - 1 + 2)$$
$$= 4$$



Another characterization of flow value

Lemma. For any flow f and any cut (S, T), we have $|f| = f(S, T)$.

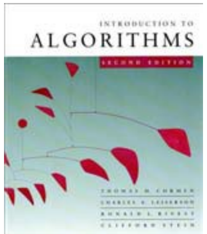
The *value* of the flow equals the flow *across the cut*.



Another characterization of flow value

Lemma. For any flow f and any cut (S, T), we have $|f| = f(S, T)$.

Proof. $f(S, T) =$

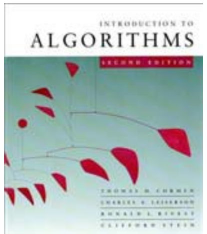


Another characterization of flow value

Lemma. For any flow f and any cut (S, T), we have $|f| = f(S, T)$.

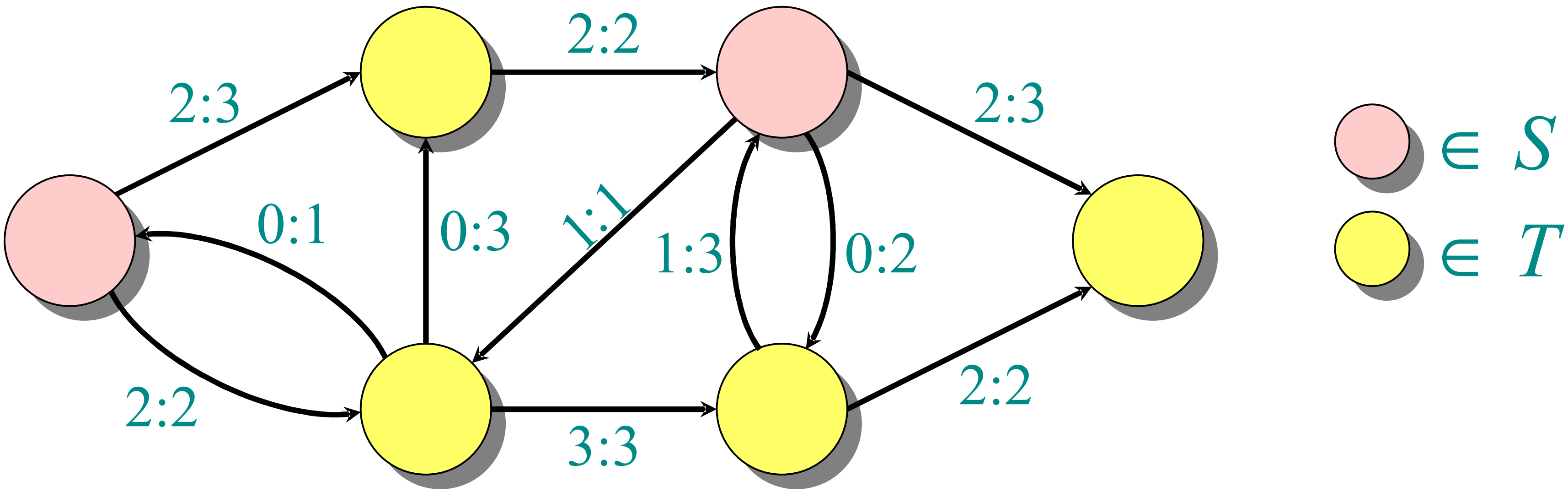
Proof.

$$
\begin{aligned}
f(S, T) &= f(S, V) - f(S, S) && T = V - S \\
&= f(S, V) && f(S, S) = 0 \\
&= f(s, V) + f(S - s, V) && S = \{s\} \cup S - \{s\} \\
&= f(s, V) && f(u, V) = 0 \text{ for } u \in S - \{s\} \\
&= |f|. \quad \blacksquare
\end{aligned}
$$

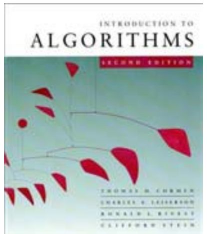


Capacity of a cut

Definition. The *capacity of a cut* (S, T) is $c(S, T)$.

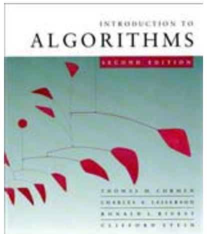


$$c(S, T) = (3 + 2) + (1 + 2 + 3)$$
$$= 11$$



Upper bound on the maximum flow value

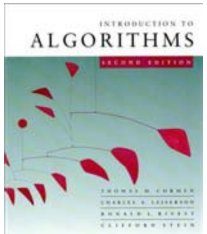
Theorem. The value of any flow is bounded above by the capacity of any cut.



Upper bound on the maximum flow value

Theorem. The value of any flow is bounded above by the capacity of any cut.

Proof. $|f| =$



Upper bound on the maximum flow value

Theorem. The value of any flow is bounded above by the capacity of any cut.

Proof.

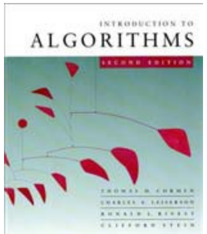
$$
\begin{aligned}
|f| &= f(S,T) && \textit{flow value = flow across cut} \\
&= \sum_{u \in S} \sum_{v \in T} f(u,v) && \textit{flow across cut} \\
&\le \sum_{u \in S} \sum_{v \in T} c(u,v) && \textit{capacity of the cut} \\
&= c(S,T) . \quad \square
\end{aligned}
$$

Residual network

Definition. Let f be a flow on $G = (V, E)$. The *residual network* $G_f(V, E_f)$ is the graph with strictly positive *residual capacities*

$$c_f(u, v) = c(u, v) - f(u, v) > 0.$$

Edges in E_f admit more flow.



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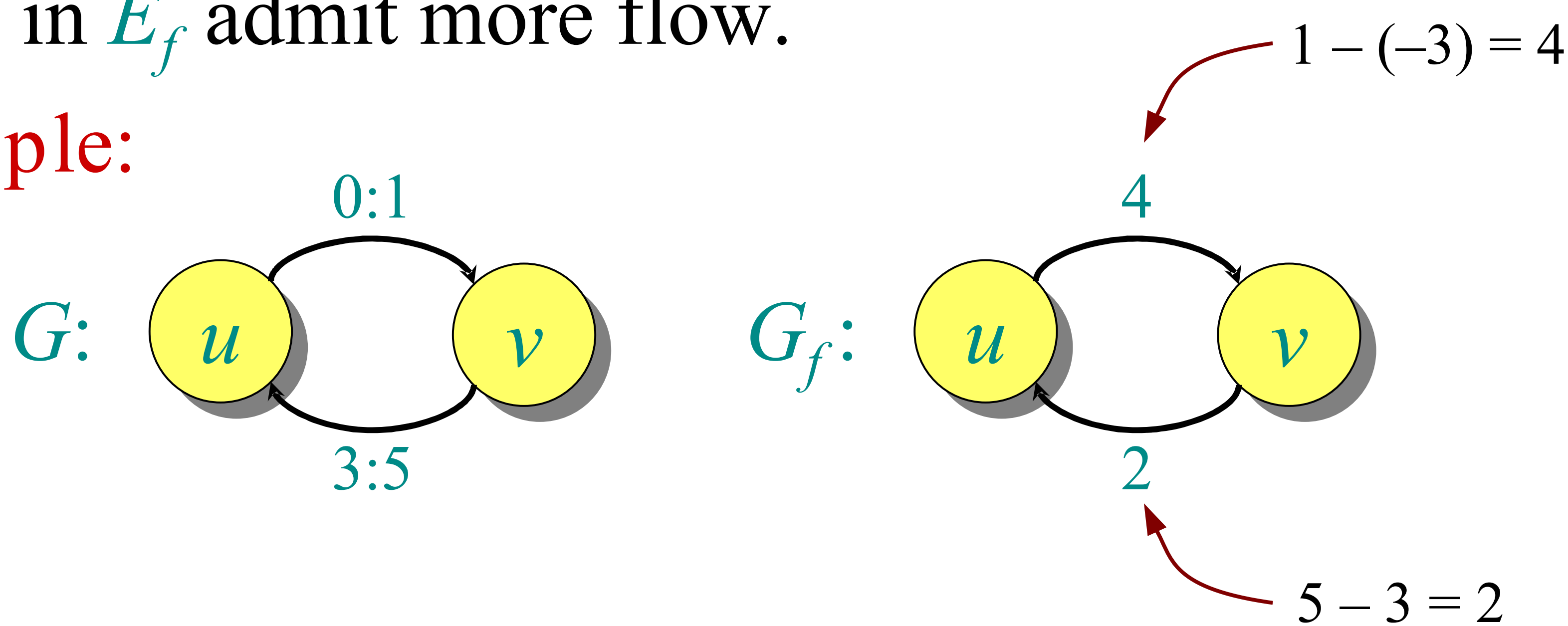
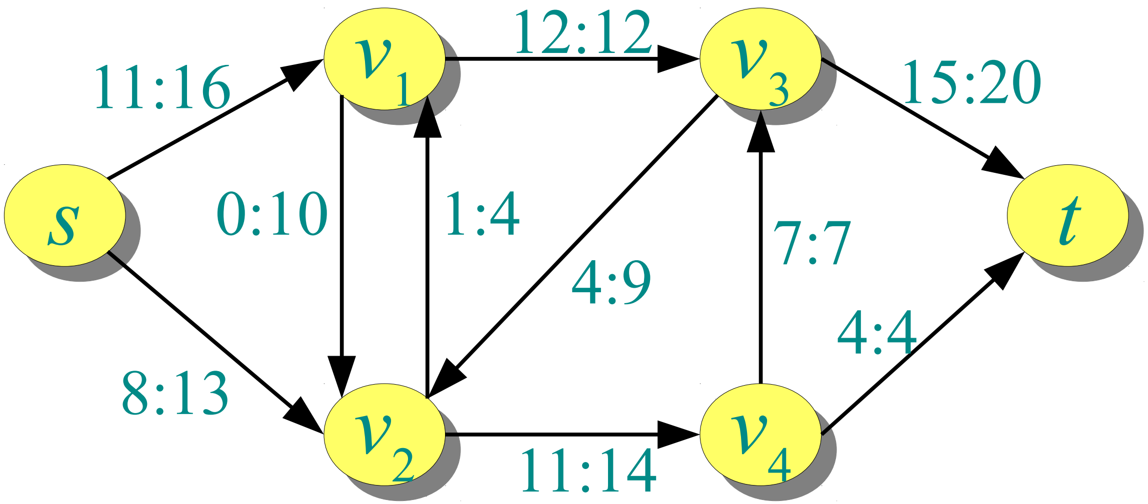
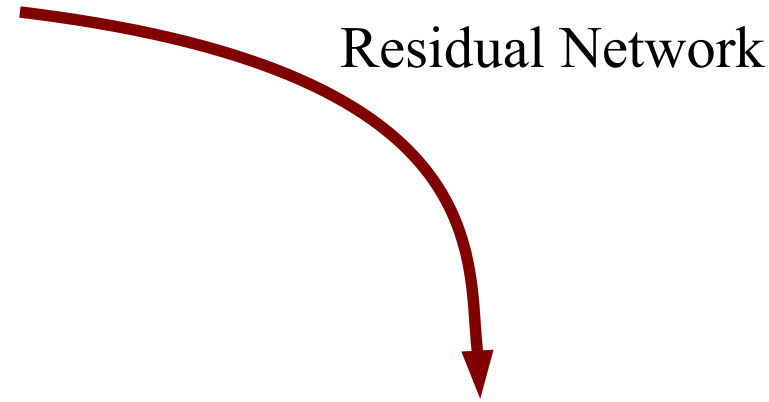
Example:



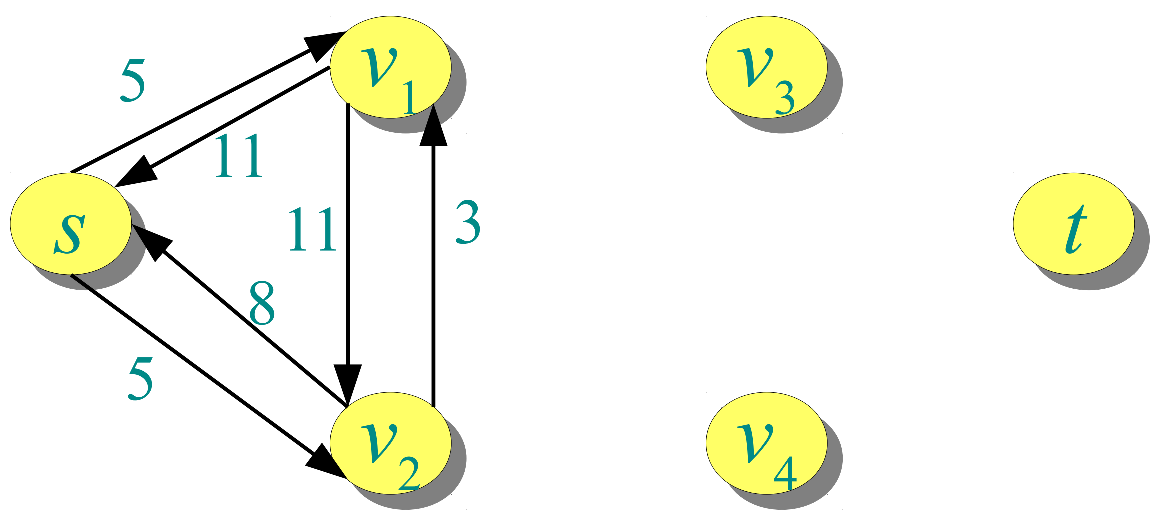
Residual Network





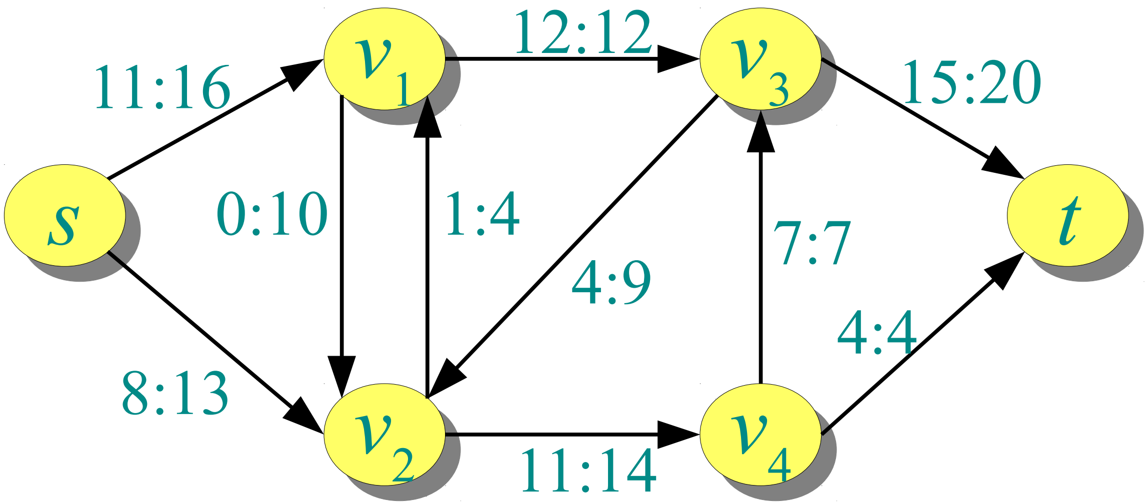
G

$$c_f(u,v) = c(u,v) - f(u,v)$$

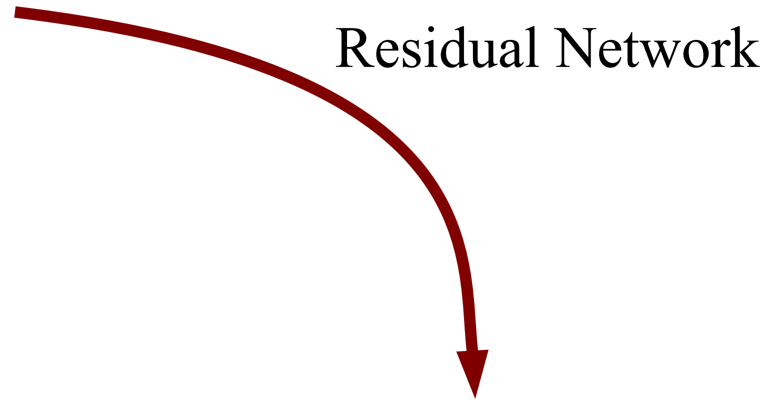


G_f

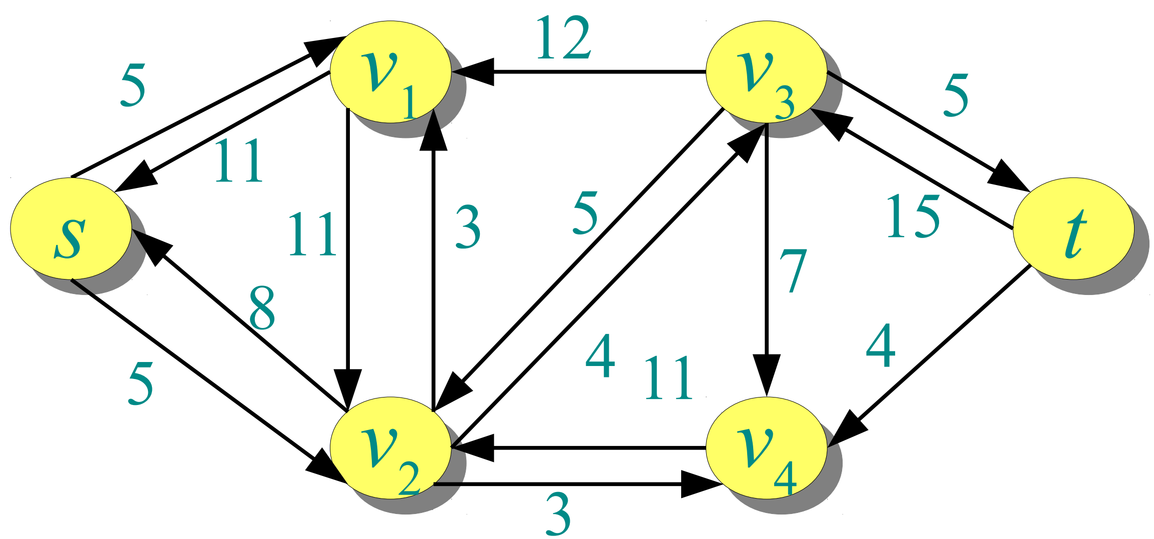
Residual Network



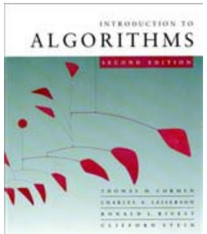
G



$$c_f(u,v) = c(u,v) - f(u,v)$$



G_f



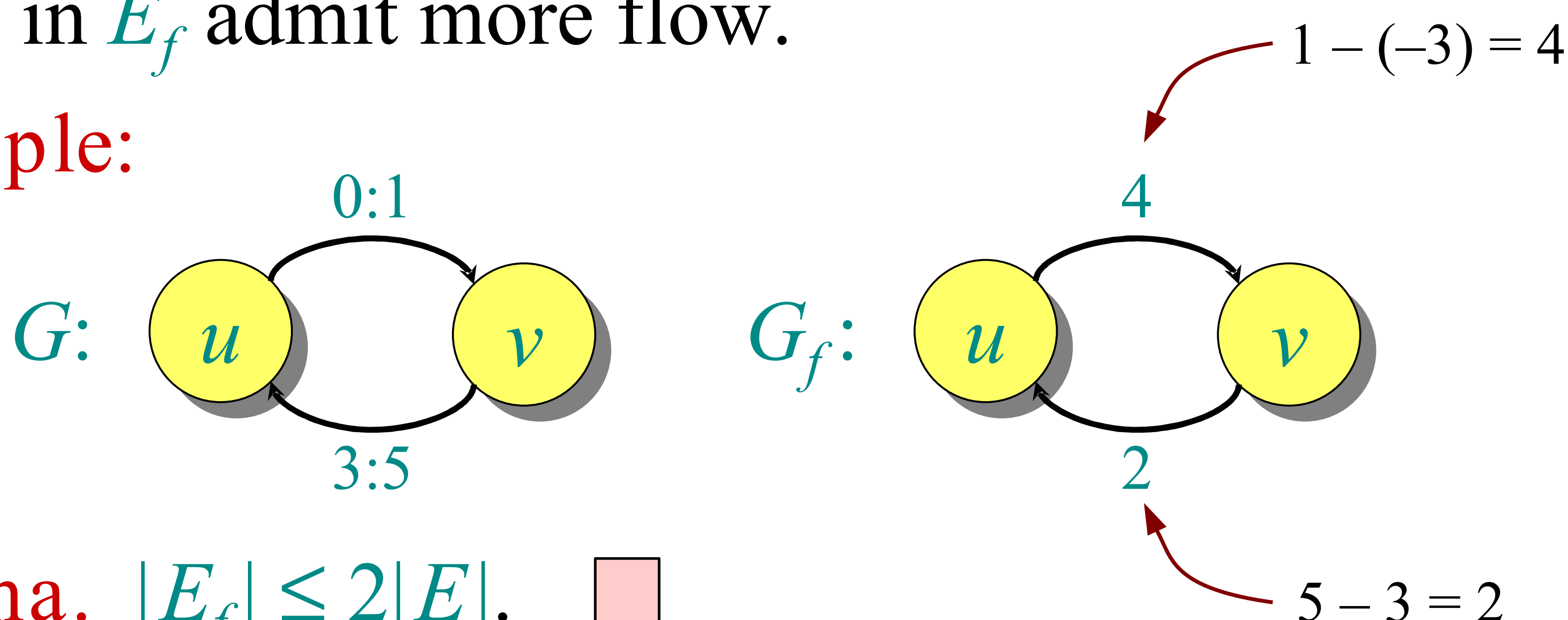
Residual network

Definition. Let f be a flow on $G = (V, E)$. The *residual network* $G_f(V, E_f)$ is the graph with strictly positive *residual capacities*

$$c_f(u, v) = c(u, v) - f(u, v) > 0.$$

Edges in E_f admit more flow.

Example:



Lemma. $|E_f| \leq 2|E|$. ■

Flow Addition

Lemma. Let $G = (V, E)$ be a flow network with source s and sink t. Let f be a flow in G. Let G_f be the residual network of G induced by f, and let f' be a flow in G_f. Then the flow sum $f + f'$ defined by $(f + f')(u, v) = f(u, v) + f'(u, v)$ is a flow in G with value $|f + f'| = |f| + |f'|$.

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Proof. Skew Symmetry:

$$\begin{aligned}(f + f')(u, v) &= f(u, v) + f'(u, v) \\ &= -f(v, u) - f'(v, u) \\ &= -(f(v, u) + f'(v, u)) \\ &= -(f + f')(v, u) \qquad \square\end{aligned}$$

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Proof. Capacity Constraints:

$$
\begin{aligned}
(f + f')(u, v) &= f(u, v) + f'(u, v) \\
&\le f(u, v) + c_f(u, v) && f'(u,v) \le c_f(u,v) \\
&= f(u, v) + c(u, v) - f(u, v) && c_f(u,v) = c(u,v) - f(u,v) \\
&= c(u, v) \quad \square
\end{aligned}
$$

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Proof. Flow Conservation:

$$\begin{aligned} \textstyle\sum_{v\in V}(f + f')(u, v) &= \textstyle\sum_{v\in V} (f(u, v) + f'(u, v)) \\ &= \textstyle\sum_{v\in V} f(u, v) + \textstyle\sum_{v\in V} f'(u, v) \\ &= 0 + 0 \\ &= 0 \qquad \square \end{aligned}$$

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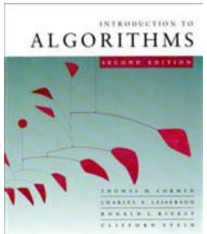
Proof. Flow Value:

$$
\begin{aligned}
|f + f'| &= \textstyle\sum_{v \in V} (f + f')(s, v) && \textit{definition} \\
&= \textstyle\sum_{v \in V} (f(s, v) + f'(s, v)) \\
&= \textstyle\sum_{v \in V} f(s, v) + \sum_{v \in V} f'(s, v)) \\
&= |f| + |f'| \quad \square
\end{aligned}
$$



Augmenting paths

Definition. A *simple* path from s to t in G_f is an *augmenting path* in G with respect to f. The flow value can be increased along an augmenting path p by $c_f(p) = \min_{(u,v) \in p} \{c_f(u,v)\}$.

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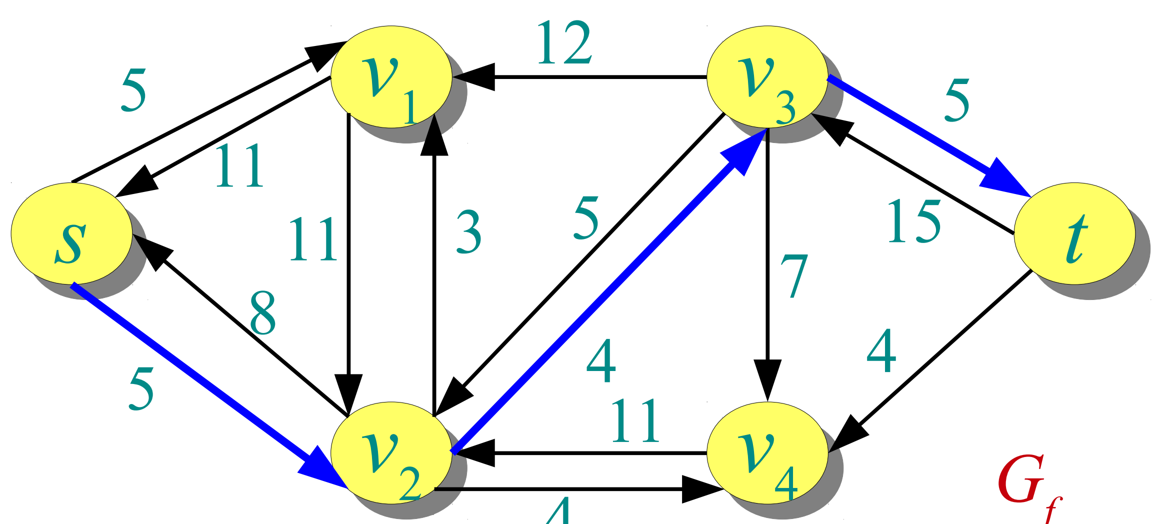
$c_f(p) = 4$



Augmenting Flow

Lemma. Let $G = (V, E)$ be a flow network with source s and sink t. Let f be a flow in G. Let G_f be the residual network of G induced by f, and let p be an *augmenting path* in G_f. Define the function $f_p : V \times V \rightarrow \mathrm{R}$ as

$$f_p(u, v) = \begin{cases} c_f(p) & \text{if } (u,v) \text{ on } p \\ -c_f(p) & \text{if } (v,u) \text{ on } p \\ 0 & \text{otherwise} \end{cases}$$

Then f_p is a flow with $|f_p| = c_f(p) > 0$. □

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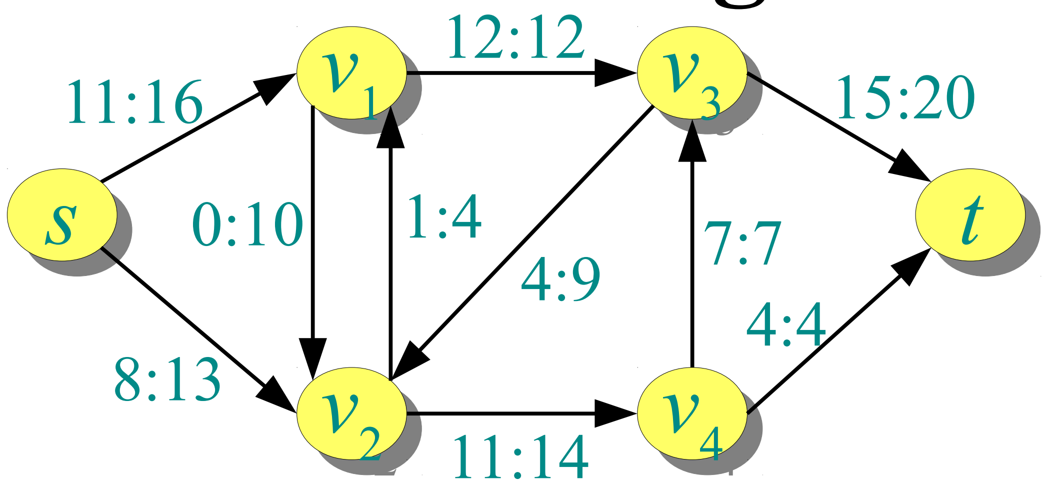
Then f_p is a flow with $|f_p| = c_f(p) > 0$. □

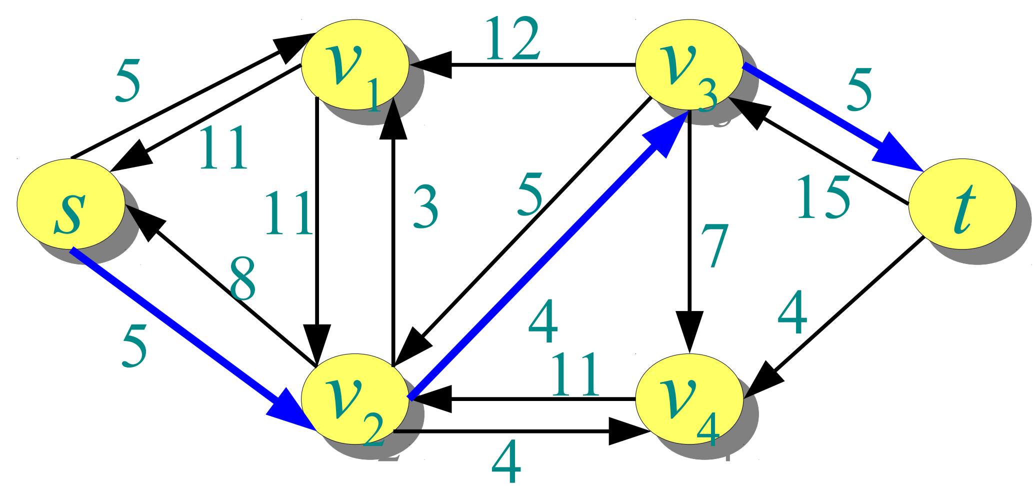
Corollary. Define the function $f' : V \times V \to \mathbb{R}$ as $f' = f + f_p$. Then f' is a flow in G with value $|f'| = |f| + |f_p| > |f|$. □

Augmenting Paths



G

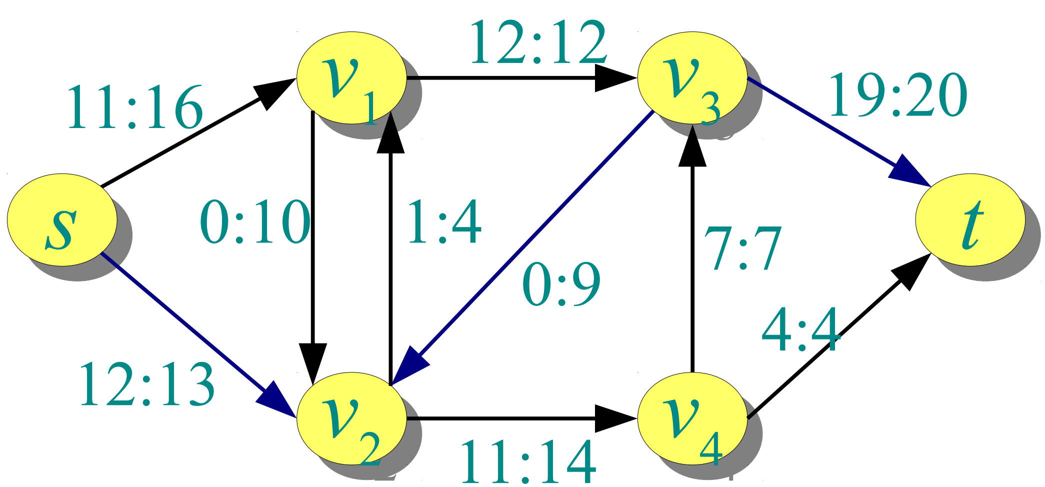
Residual Network



G_f

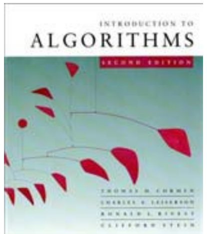
$$f'(u,v) = f(u,v) + f_p(u,v)$$

$$c_f(p) = 4$$



G'

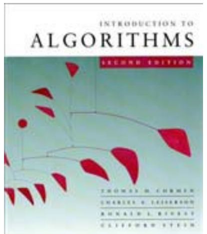
Augmenting Flow



Max-flow, min-cut theorem

Theorem. The following are equivalent:

1. f is a maximum flow.
2. G_f contains no augmenting paths.
3. $|f| = c(S, T)$ for some cut (S, T) of G.



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Proof (and algorithms). Next time. □

Recap

- Flow Networks
- Maximum Flow Problem
- Flow Notation
- Properties of Flow
- Cuts
- Residual Networks
- Augmented Paths