CMP205: Computer Graphics

Lecture 3: Transformations 2D

Mohamed Alaa El-Dien Aly
Computer Engineering Department
Cairo University
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2D Transformations

- Scale
- Shear
- Rotation
- Reflection

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_{11}x + a_{12}y \\ a_{21}x + a_{22}y \end{bmatrix}$$

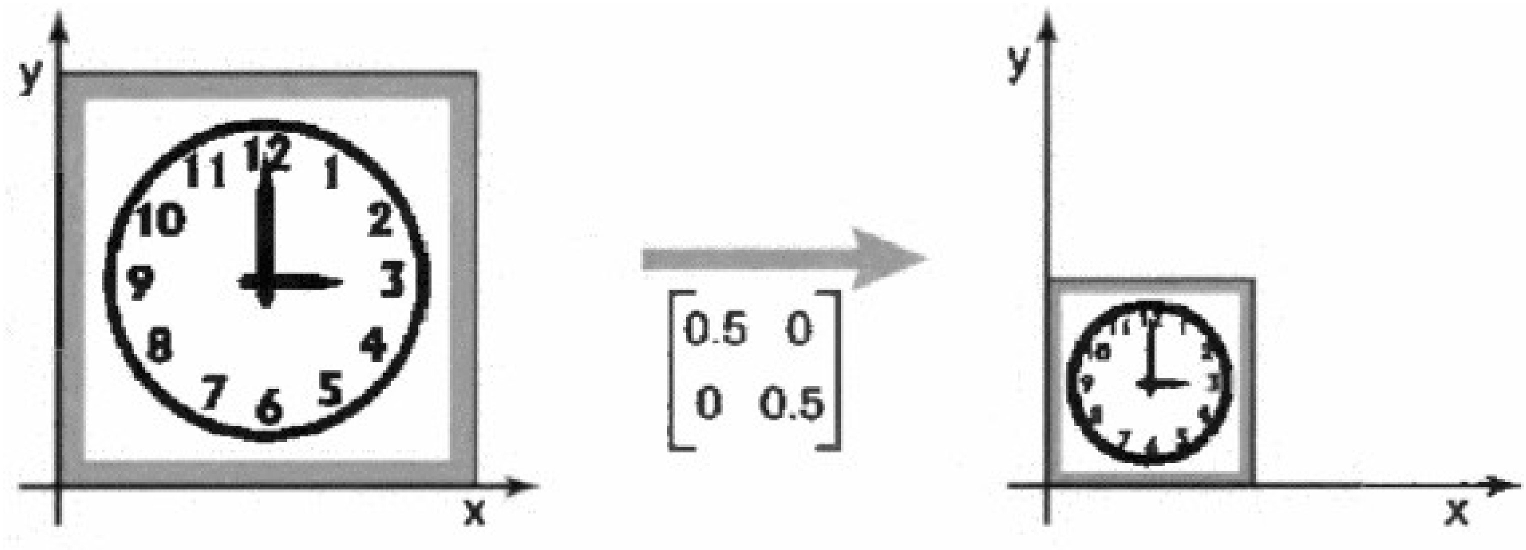
Scale

$$\text{Scale}(s_x, s_y) = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$

$$\begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} s_x x \\ s_y y \end{bmatrix}$$

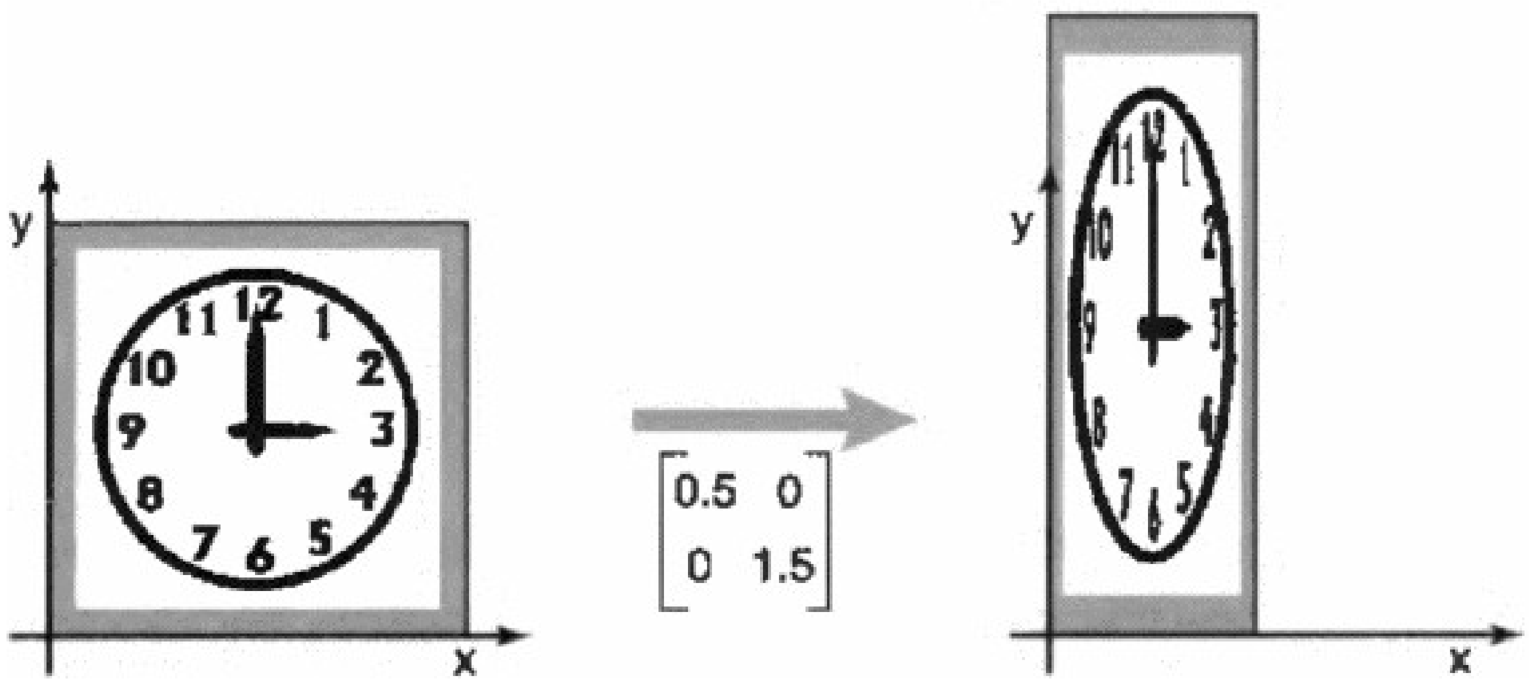
Scale

- Uniform Scale



Scale

- Nonuniform Scale



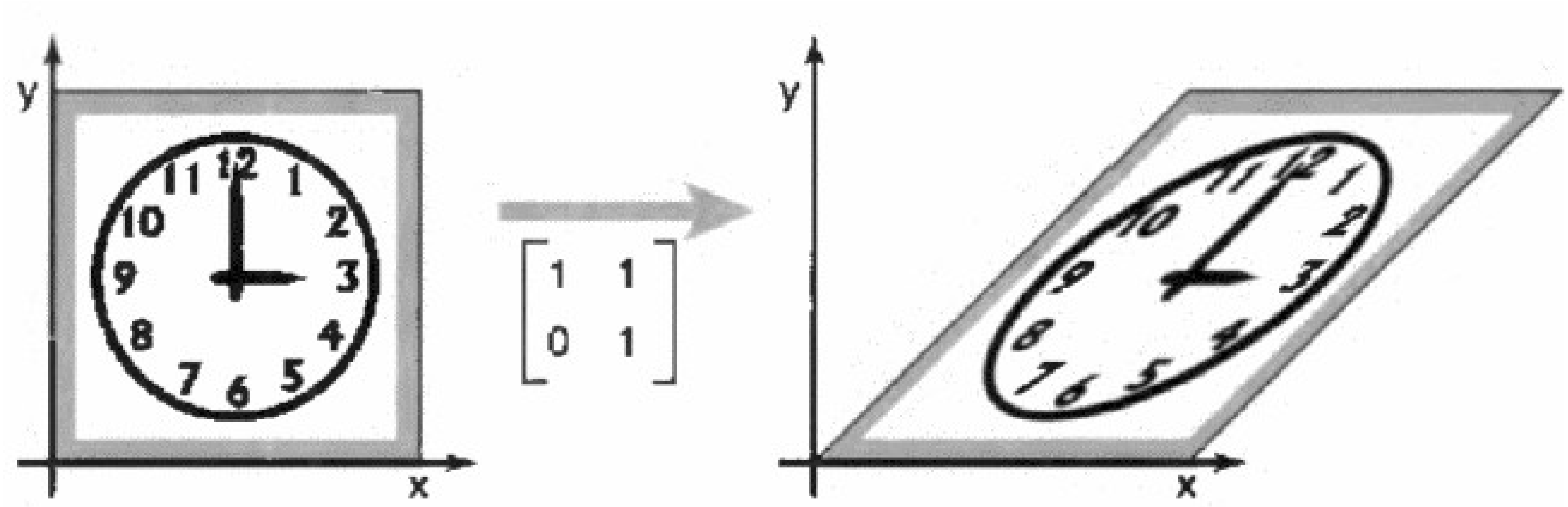
Shearing

$$\text{shear-x}(s) = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$$

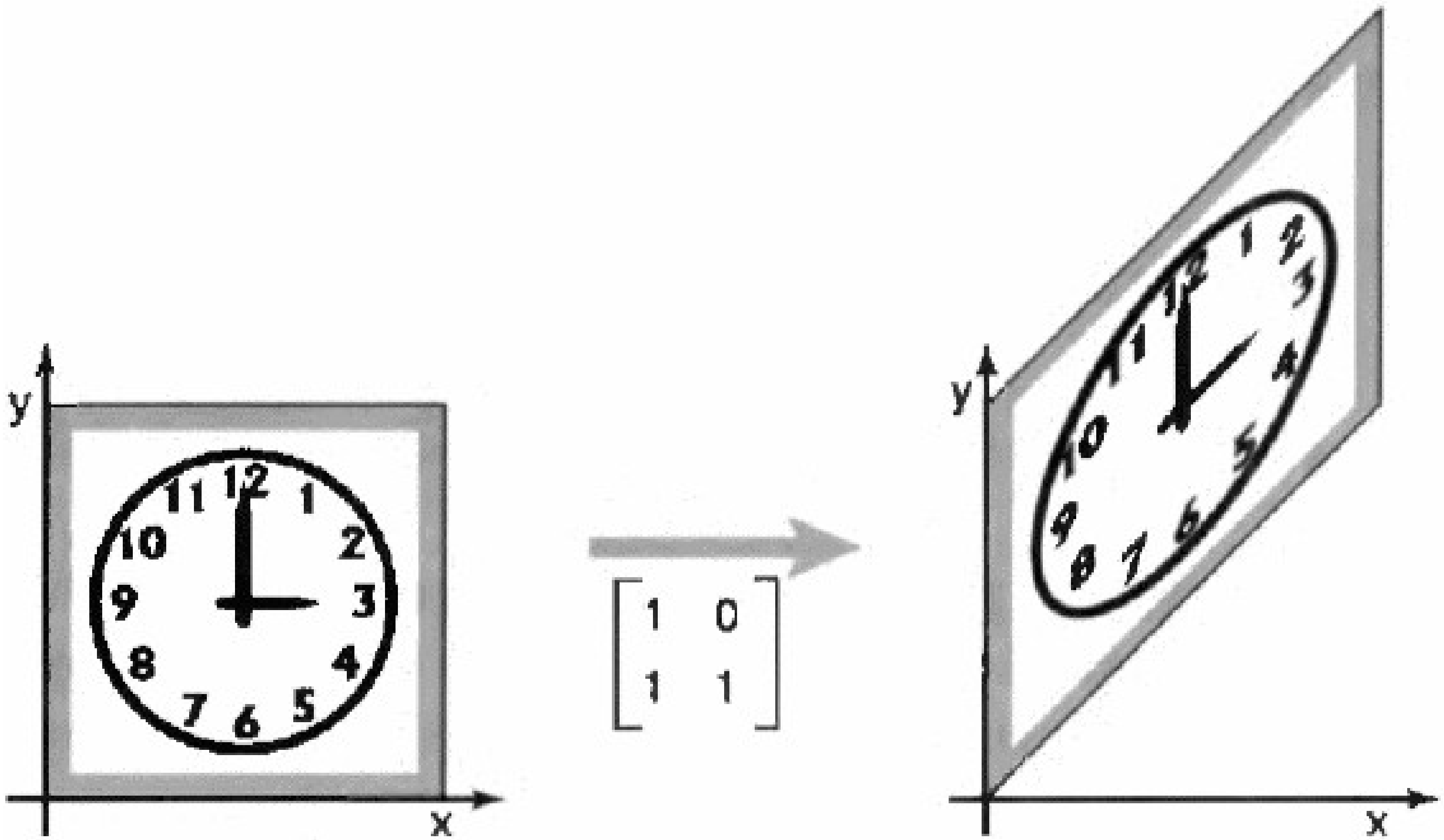
$$\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + sy \\ y \end{bmatrix}$$

$$\text{shear-y}(s) = \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix}$$

Shearing

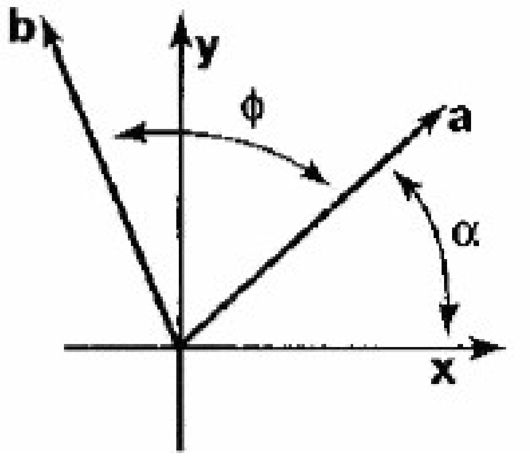


Shearing



Rotation

$$x_a = r\cos\alpha \qquad x_b = r\cos(\alpha+\phi)$$
$$y_a = r\sin\alpha \qquad y_b = r\sin(\alpha+\phi)$$



and we know that:

$$\cos(\alpha+\phi) = \cos\alpha\cos\phi - \sin\alpha\sin\phi$$
$$\sin(\alpha+\phi) = \cos\alpha\sin\phi + \sin\alpha\cos\phi$$

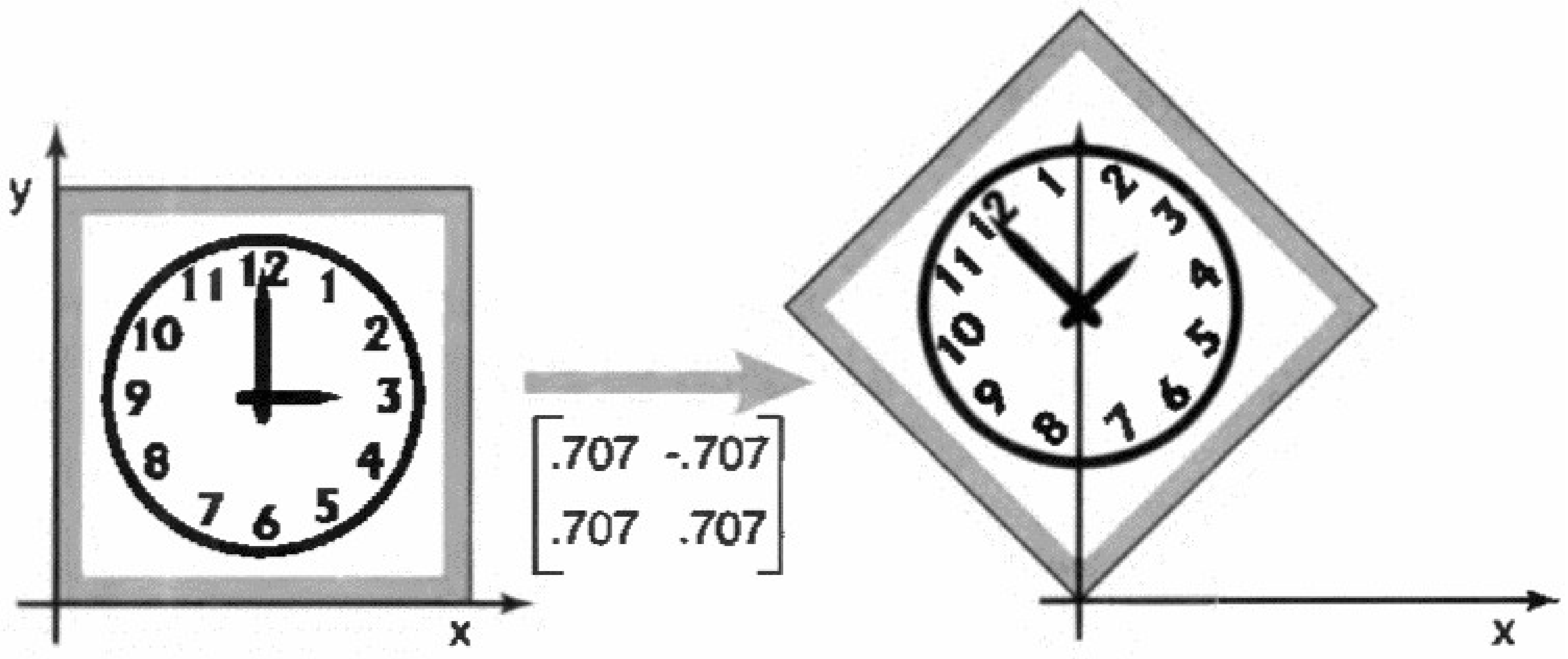
$$x_b = x_a\cos\phi - y_a\sin\phi$$
$$y_b = x_a\sin\phi + y_a\cos\phi$$

Rotation

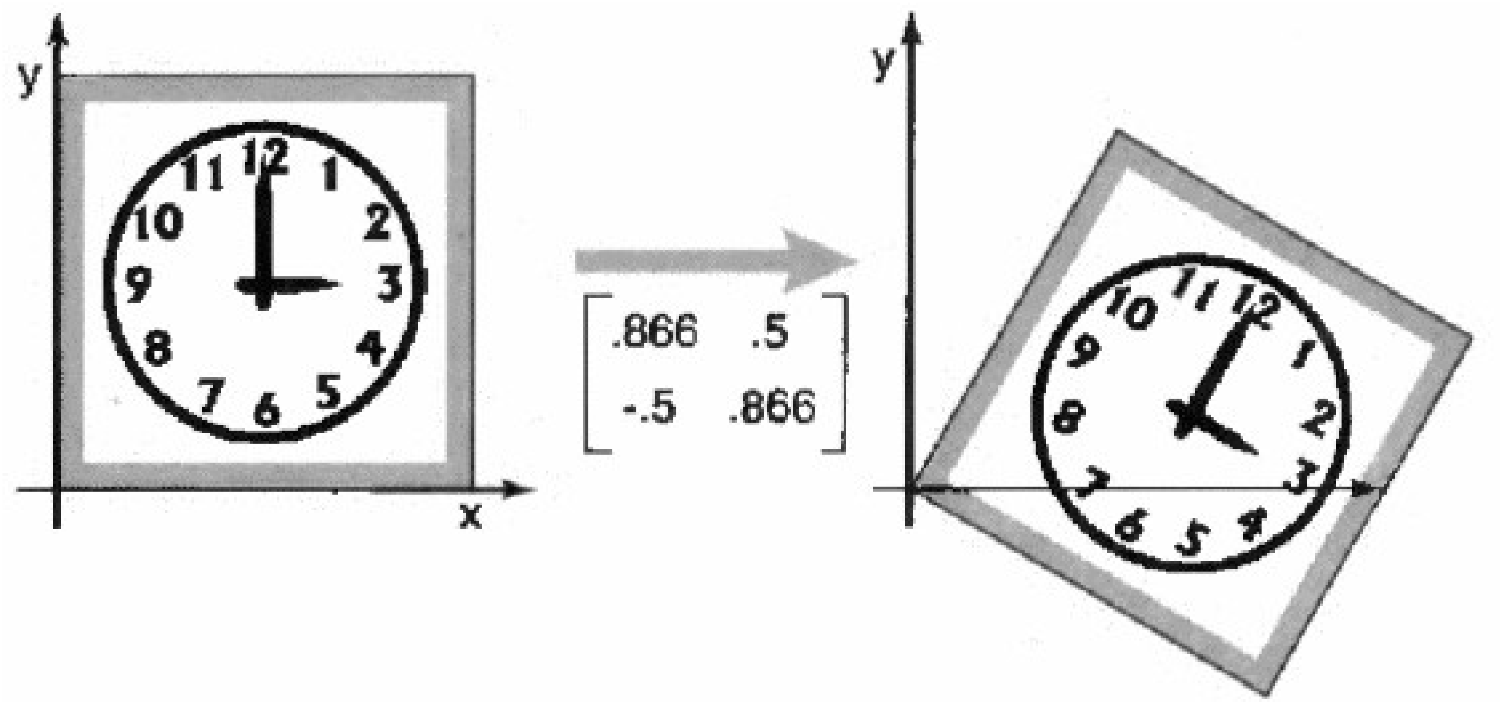
$$\text{rotate}(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

Property?

Rotation



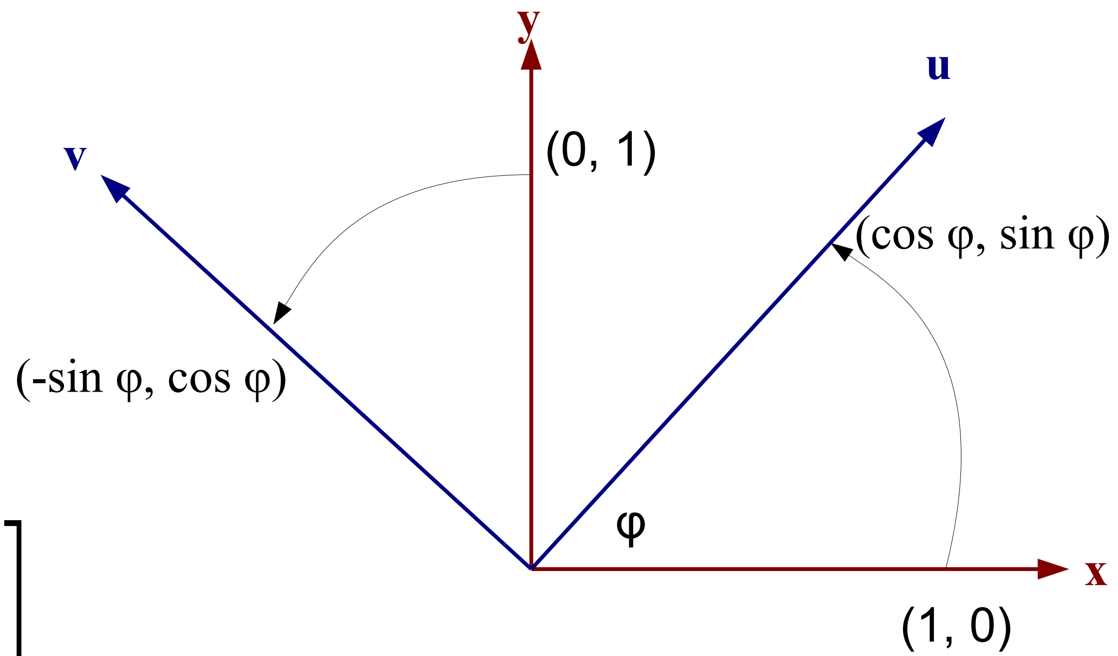
Rotation



Transformation Vs Coordinate Change

$$R = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

$$\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix}$$



$$\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin\phi \\ \cos\phi \end{bmatrix}$$

Transformation Vs Coordinate Change

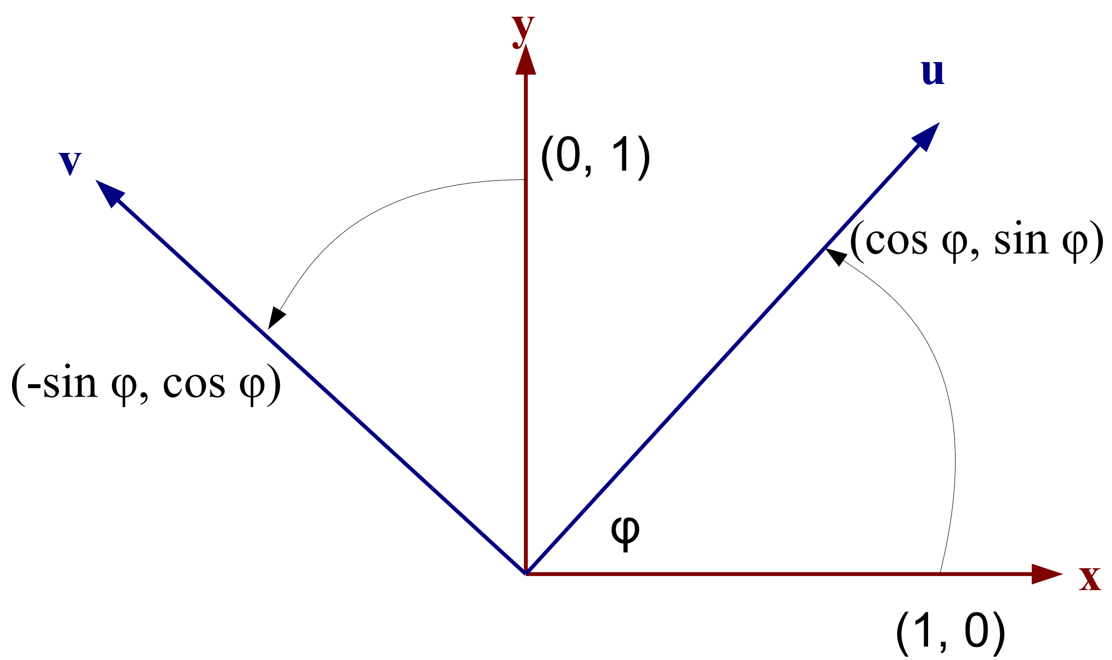
Transformation:

$$p' = R\,p$$

$$R = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

Coordinate Change:

$${}^{xy}p = R\,{}^{uv}p$$



R transforms points in *xy* coordinates OR transforms *uv* coordinates to *xy* coordinates

What about R^T?

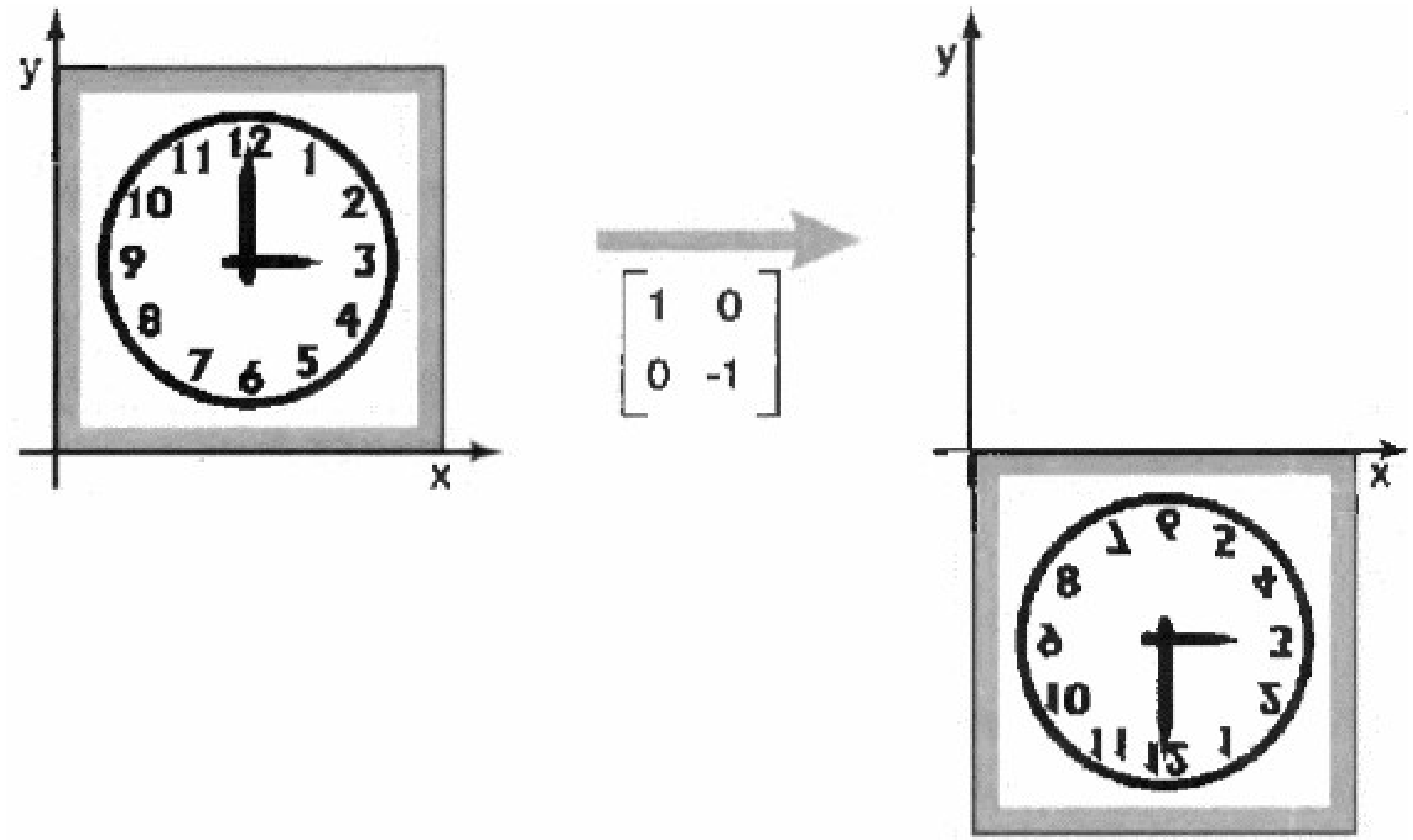
Reflection

$$\text{reflect-x} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

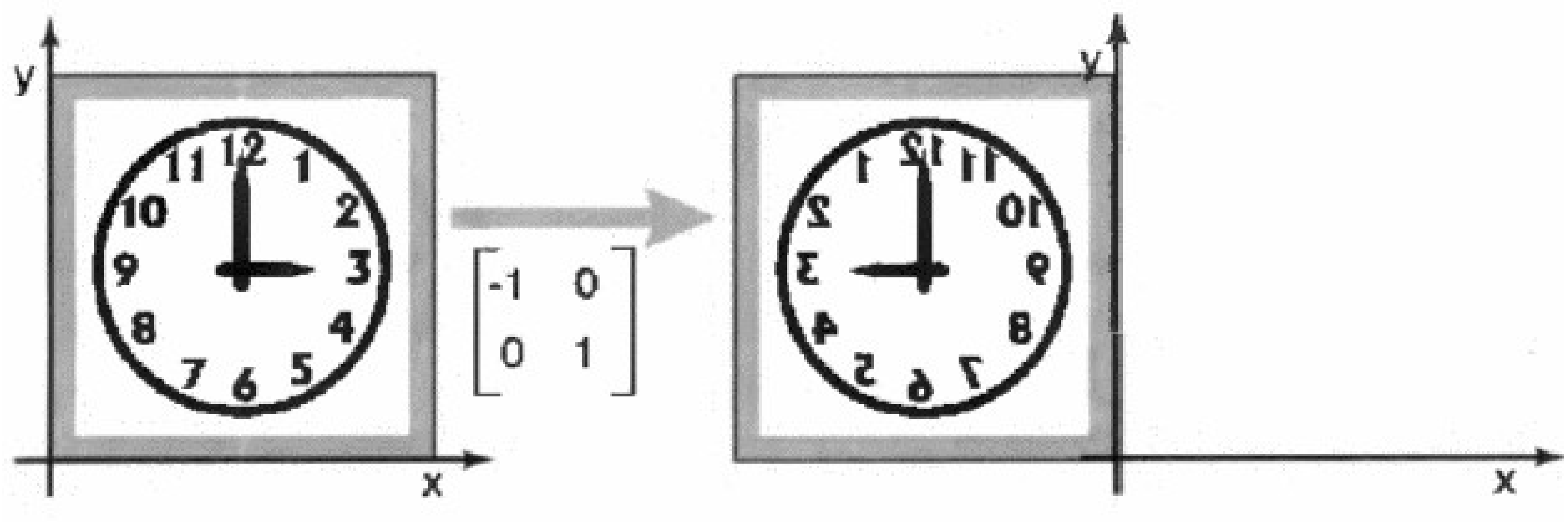
$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$$

$$\text{reflect-y} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

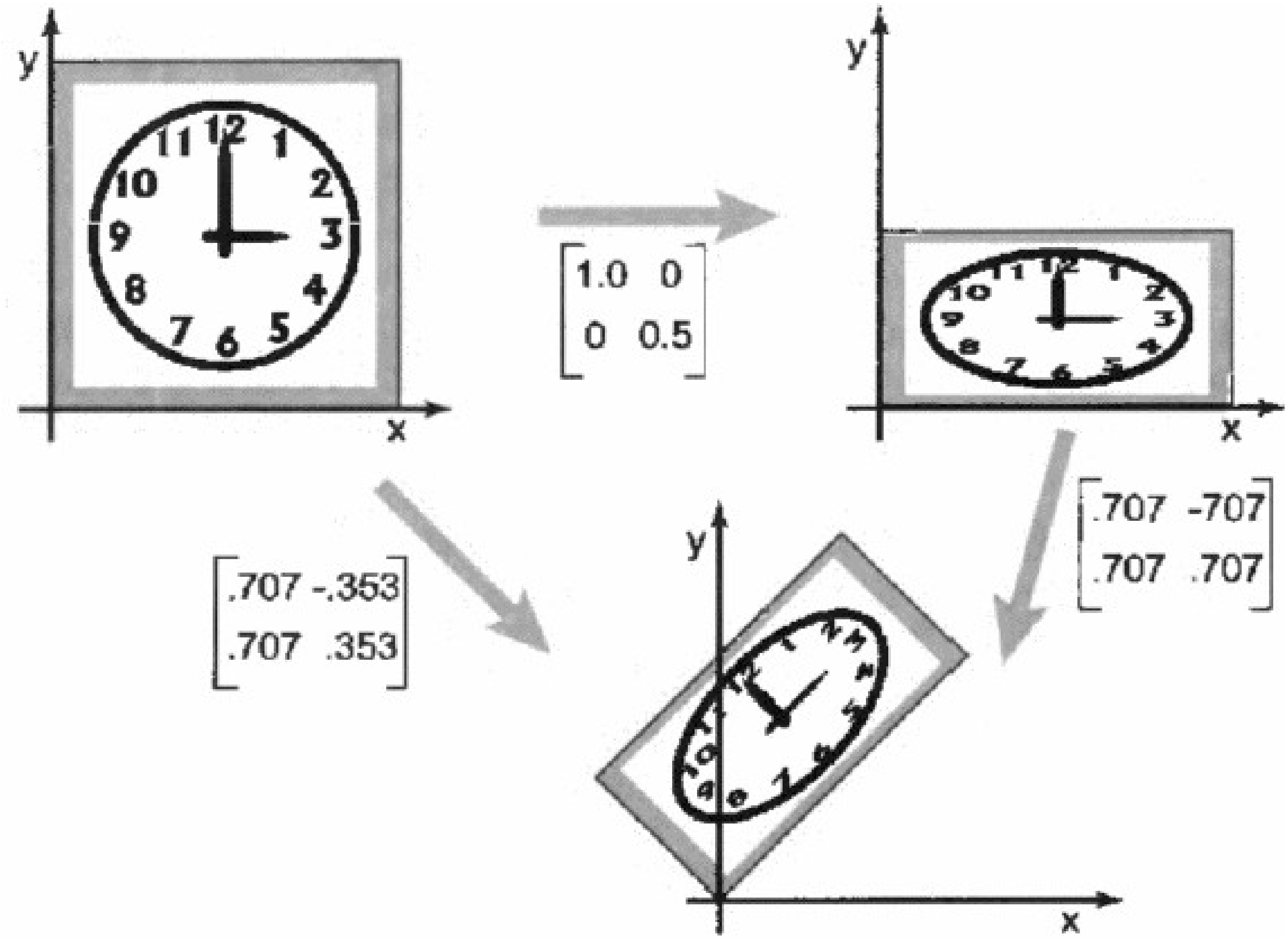
Reflection



Reflection



Composition of 2D Transforms



$$v_2 = S\, v_1$$

$$v_3 = R\, v_2$$

$$\downarrow$$

$$v_3 = (R\, S)\, v_1$$

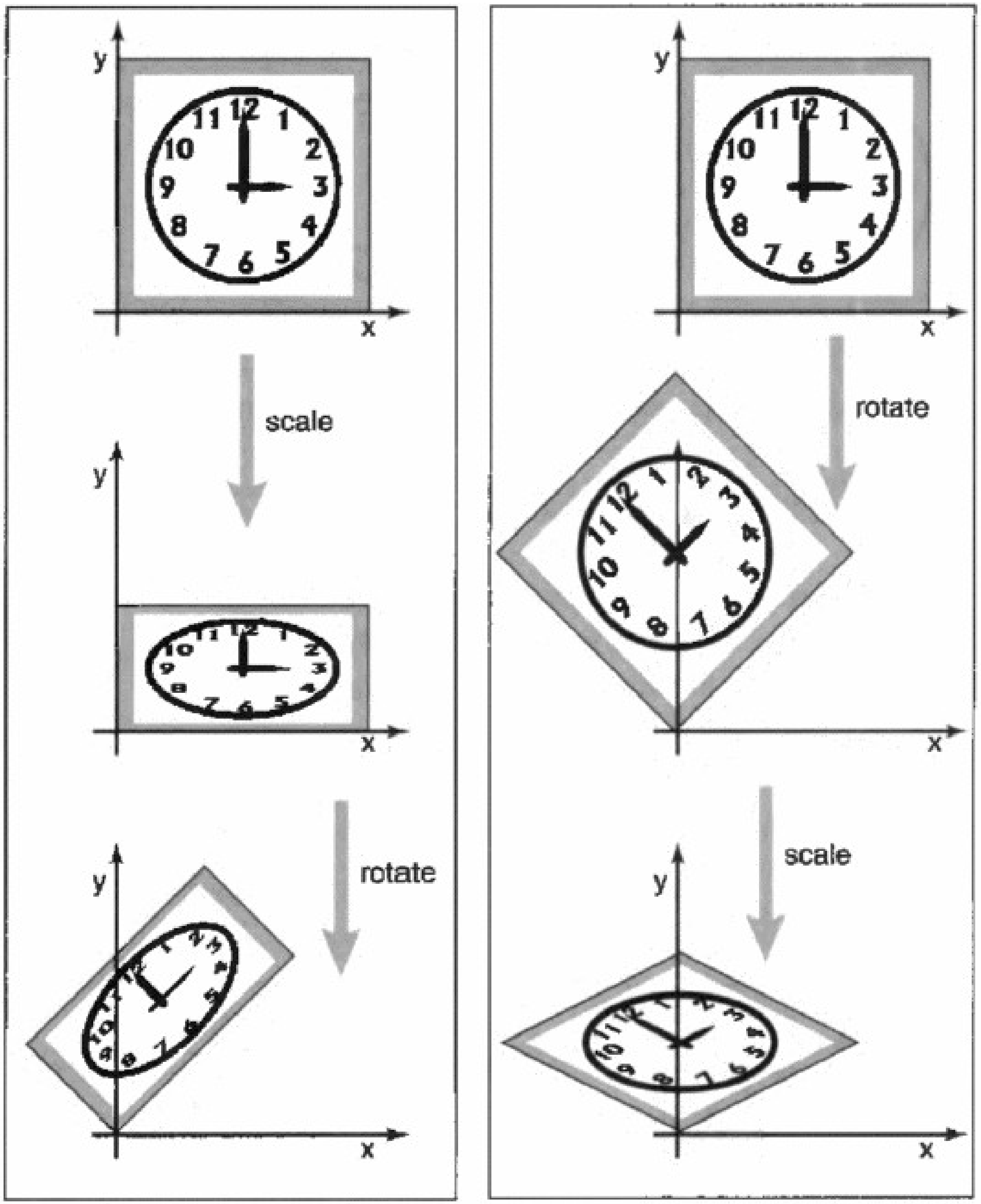
$$v_3 = M\, v_1$$

where:

$$M = R\, S$$

Composition of 2D Transforms

$$RS \neq SR$$



Decomposition of 2D Transforms

- Any 2D Transform matrix can be decomposed:
  - Rotation → Scale → Rotation
  - Using SVD (Singular Value Decomposition)
- Symmetric Transforms
  - Rotation → Scale → Inverse Rotation
  - Using Eigenvalue decomposition
- Only basic operations
  - rotation
  - translation

Eigenvalue Decomposition

$M v = \lambda v$

v is an eigenvector, and
λ is corresponding eigenvalue

In matrix form:
$M V = V \Lambda$

If M is symmetric
$M = R \Lambda R^T$

where
R is orthonormal, and
Λ is diagonal

Singular Value Decomposition

$$M = USV$$

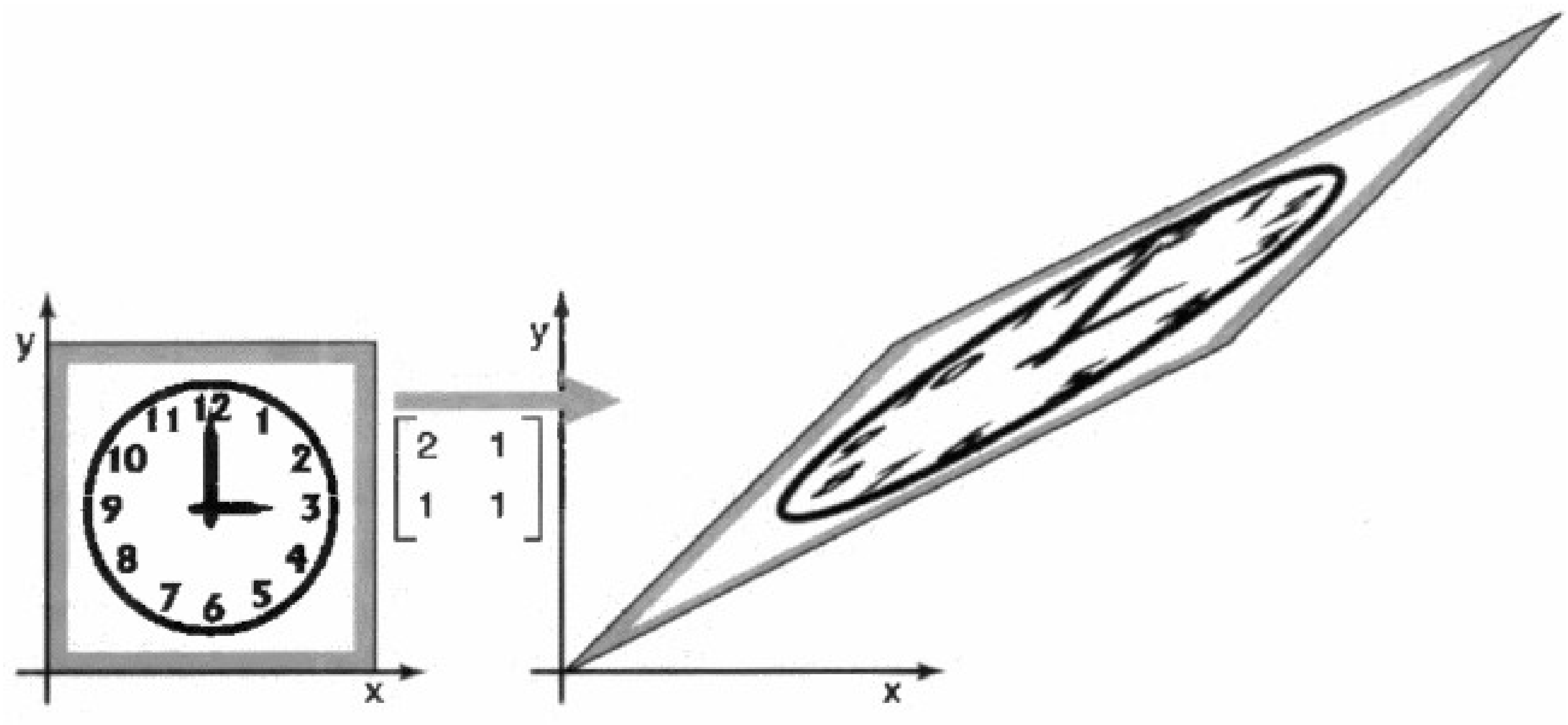
$where$

U and V are orthonormal, and

S is diagonal

Decomposition of 2D Transforms

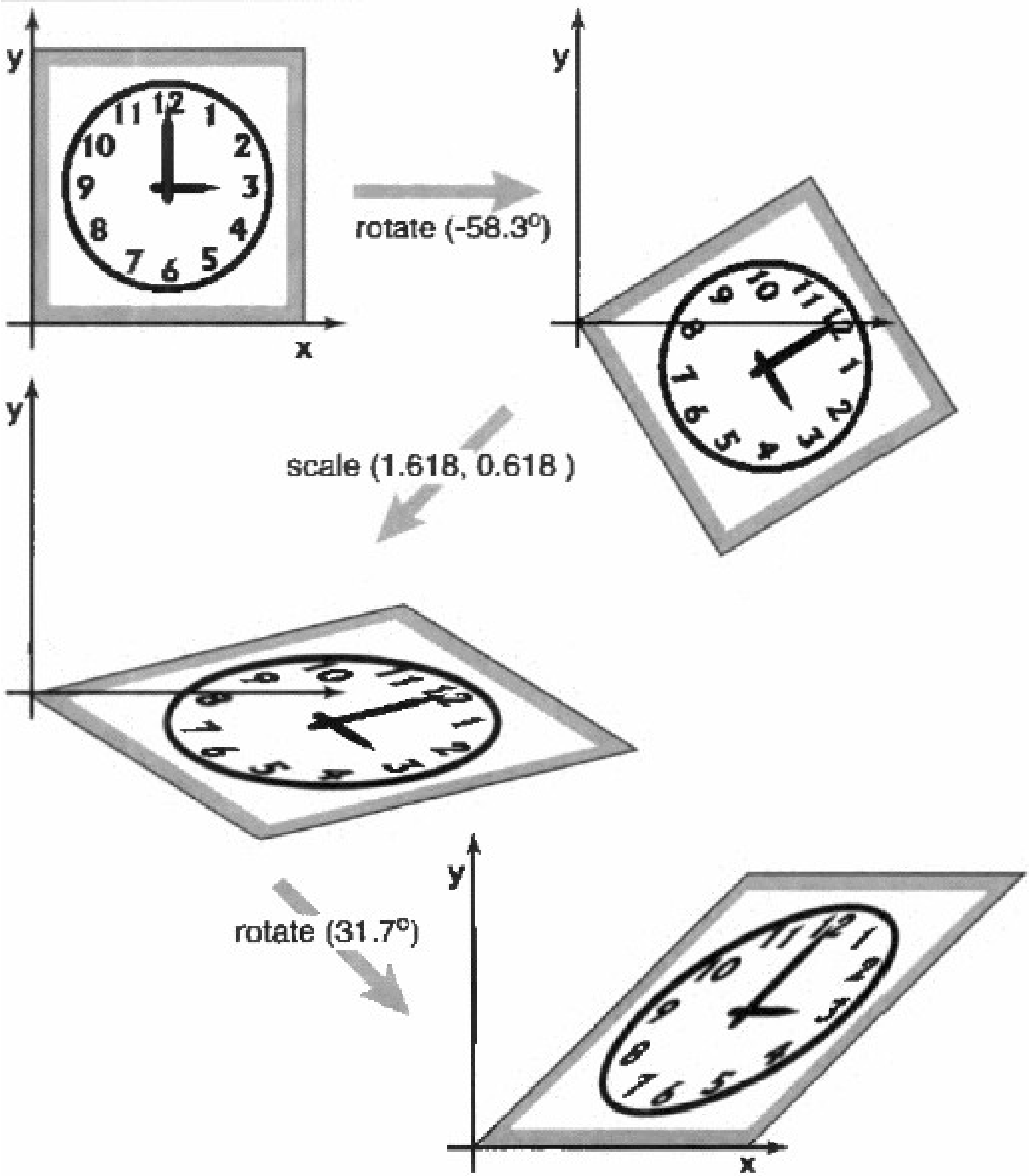
Eigenvalue Decomposition $M = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$



Decomposition of 2D Transforms

SVD Decomposition

$$M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$



Recap

- 2D Transformations
- Composition
- Decomposition