CMP205: Computer Graphics

CAIRO UNIVERSITY

Lecture 4: Transformations 3D

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Fall 2012

Agenda

- 3D Transformations
- Translations
- Windowing Transforms
- Coordinate Transformations

3D Transforms

- Scaling
- Rotation
- Shear

Scaling

$$\text{scale}(s_x, s_y, s_z) = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{bmatrix}$$

Rotation

$$\text{rotate-z}(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{rotate-x}(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}$$

$$\text{rotate-y}(\phi) = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix}$$

Shear

$$\text{shear-x}(d_y, d_z) = \begin{bmatrix} 1 & d_y & d_z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Arbitrary Rotation

$$R = \begin{bmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \\ x_w & y_w & z_w \end{bmatrix} = \begin{bmatrix} u^T \\ v^T \\ w^T \end{bmatrix}$$

$$R\,R^T = I$$

$$Ru = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = x$$

R takes uvw to xyz

$$R^T x = \begin{bmatrix} x_u \\ y_u \\ z_u \end{bmatrix} = u$$

R^T takes xyz to uvw

Arbitrary Rotation

- To rotate about an arbitrary axis a
  - Create axes uvw s.t. w coincides with a
  - Change uvw to xyz using R
  - Perform the rotation in xyz around z-axis
  - Change back to uvw using R^T

$$\begin{bmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \\ x_w & y_w & z_w \end{bmatrix}$$

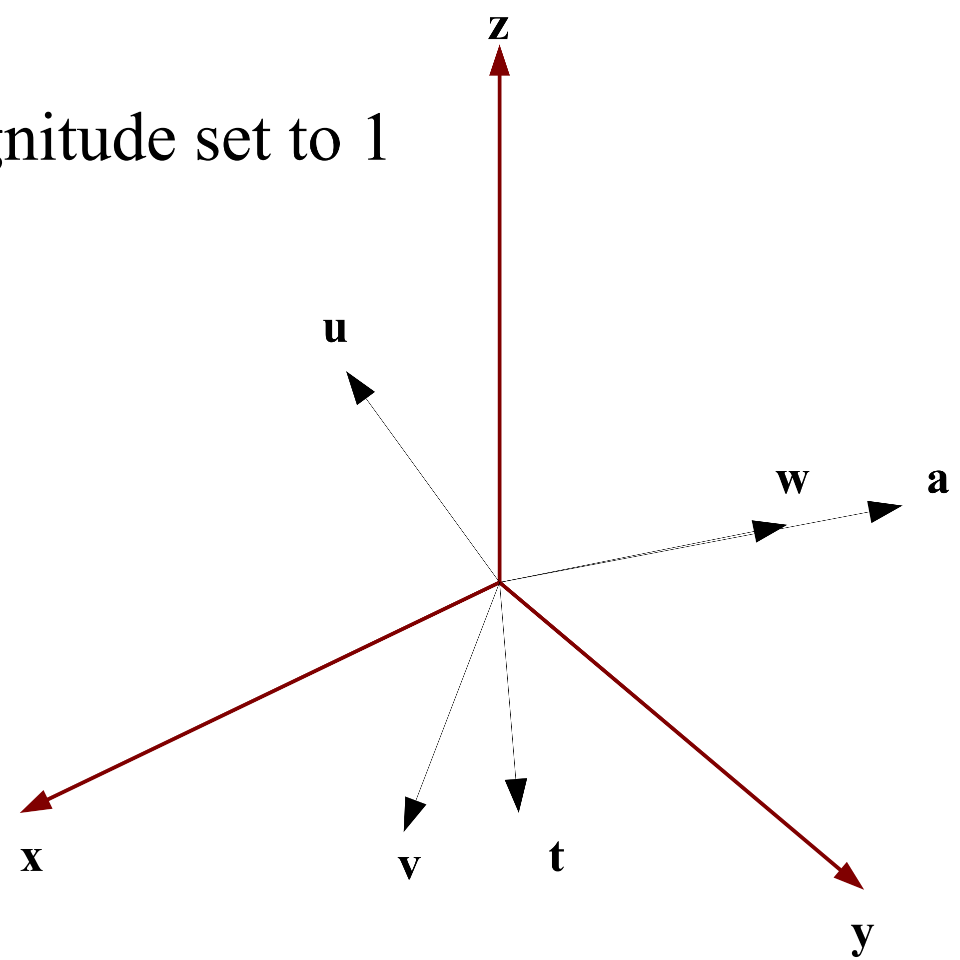
Arbitrary Rotation

$$w = \frac{a}{\|a\|}$$

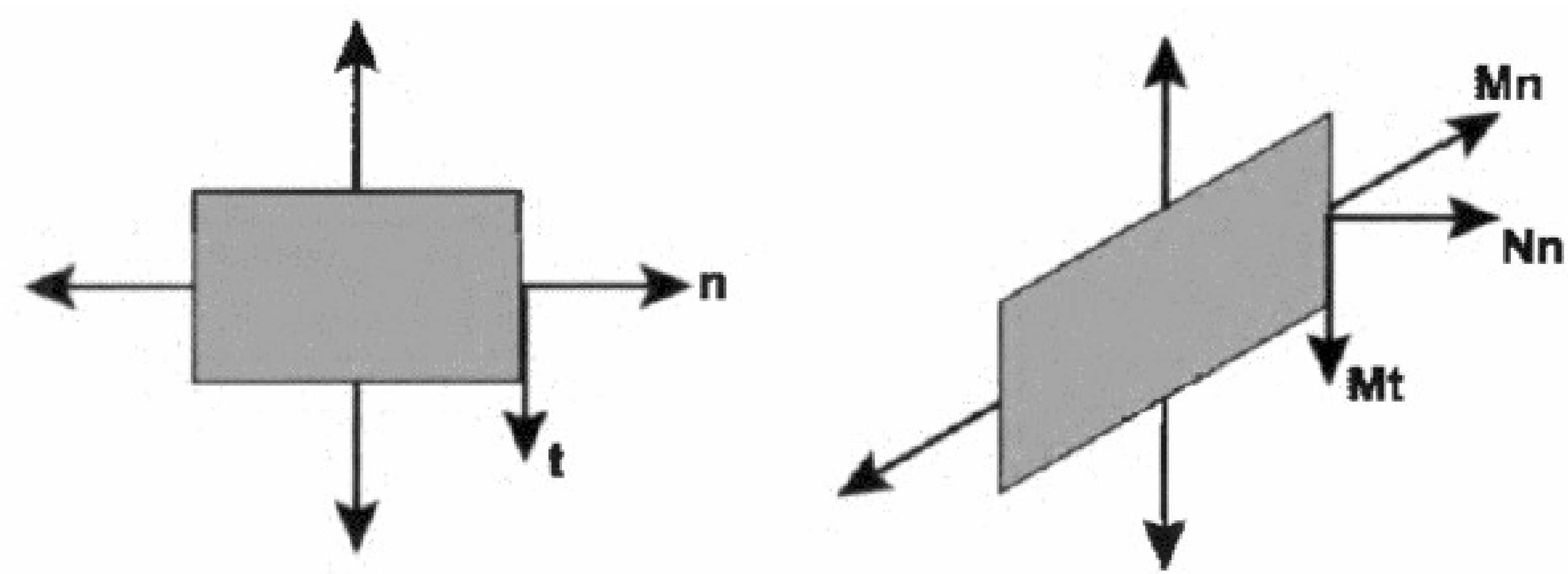
$t = w'$ i.e. w with lowest magnitude set to 1

$$u = \frac{t \times w}{\|t \times w\|}$$

$$v = w \times u$$



Transforming Normal Vectors



Mn is not normal to the surface!

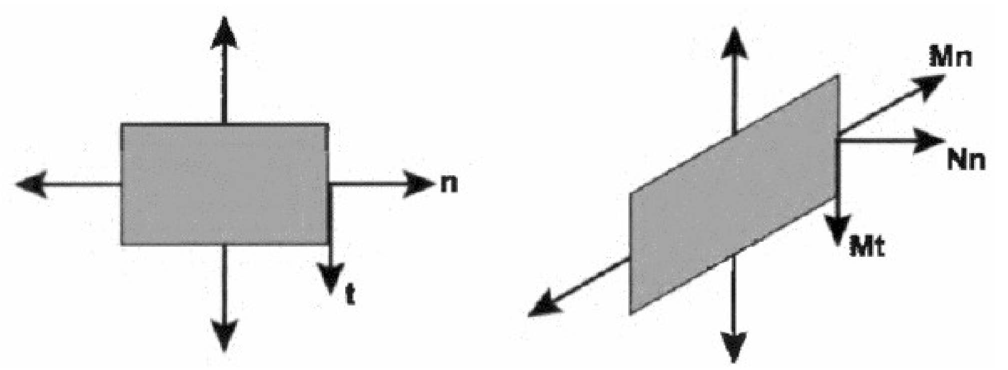
What is N?

Transforming Normal Vectors

Derivation

$$n' = N\,n \text{ and } t' = M\,t$$

$$n^T t = 0$$
$$n^T M^{-1} M\,t = 0$$
$$\left(n^T M^{-1}\right)\left(M\,t\right) = 0$$
$$\left(\left(M^{-1}\right)^T n\right)^T \left(M\,t\right) = 0$$
$$\left(n'\right)^T t' = 0$$



$$N = \left(M^{-1}\right)^T$$

2D Translations

2D Transformations

$$p' = M_{2\times 2}\, p$$

$$x' = m_{11} x + m_{12} y$$
$$y' = m_{21} x + m_{22} y$$

Translation

$$x' = x + t_x$$
$$y' = y + t_y$$

How can we represent *Translation* as Matrix Multiplication?

2D Translations

3D Shear in *XY* plane

$$\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + t_x z \\ y + t_y z \\ z \end{bmatrix}$$

Translation

$$x' = x + t_x$$
$$y' = y + t_y$$

Solution: add z=1 to 2D points

$$\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix}$$

What about vectors?

z = 0 !

Homogeneous Coordinates

Convert 2D points into 3D points

$$v = \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \tilde{v} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

2D Transformations with 3x3 matrix

$$\tilde{v}' = M\tilde{v}$$

2D Transformations

Translation

$$\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

Rotation

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$v = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Shear

$$\begin{bmatrix} 1 & s & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Scaling

$$\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Can represent any combination by a 3x3 matrix

2D Transformations

Rotation/Scale/Shear + Translation

$$\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} R_{2\times 2} & t_{2\times 1} \\ 0^T & 1 \end{bmatrix}$$

Rotation part + translation part

3D Translations

Homogeneous Coordinates

$$\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x + t_x \\ y + t_y \\ z + t_z \\ 1 \end{bmatrix}$$

3D Transformations

Translation

$$\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$v = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Can represent any combination by a 4x4 matrix

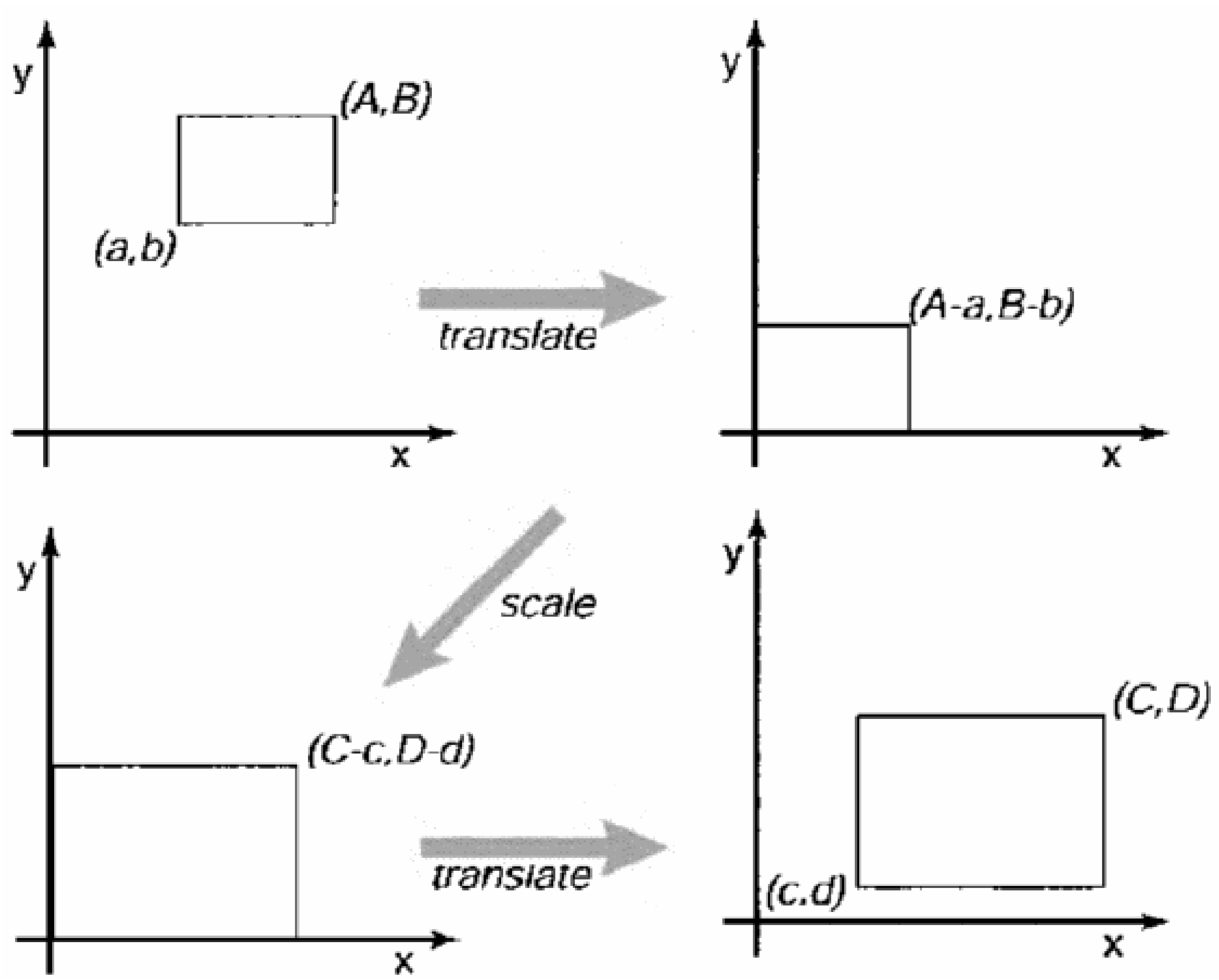
3D Transformations

Rotation/Scale/Shear + Translation

$$\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{21} & a_{22} & a_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & t_x \\ a_{21} & a_{22} & a_{23} & t_y \\ a_{21} & a_{22} & a_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} R_{3\times 3} & t_{3\times 1} \\ 0^T & 1 \end{bmatrix}$$

Rotation part + translation part

Windowing Transforms

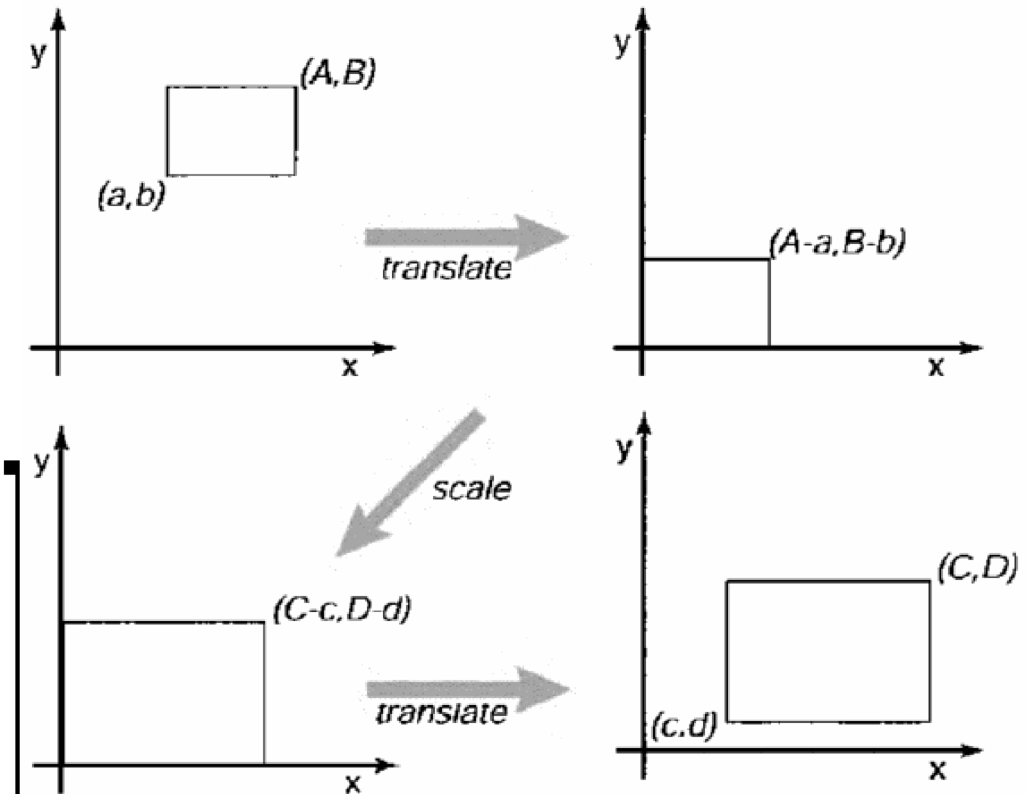


Windowing Transforms

$$\text{translate}(-a,-b)=\begin{bmatrix}1 & 0 & -a\\ 0 & 1 & -b\\ 0 & 0 & 1\end{bmatrix}$$

$$\text{scale}\left(\frac{C-c}{A-a},\frac{D-d}{B-b}\right)=\begin{bmatrix}\frac{C-c}{A-a} & 0 & 0\\ 0 & \frac{D-d}{B-b} & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\text{translate}(c,d)=\begin{bmatrix}1 & 0 & c\\ 0 & 1 & d\\ 0 & 0 & 1\end{bmatrix}$$



$$\begin{bmatrix}\frac{C-c}{A-a} & 0 & \frac{cA-Ca}{A-a}\\ 0 & \frac{D-d}{B-b} & \frac{dB-Db}{B-b}\\ 0 & 0 & 1\end{bmatrix}$$

Transformation Inverse

$$M \rightarrow M^{-1}$$

$$Rotation\ R \rightarrow R^T$$

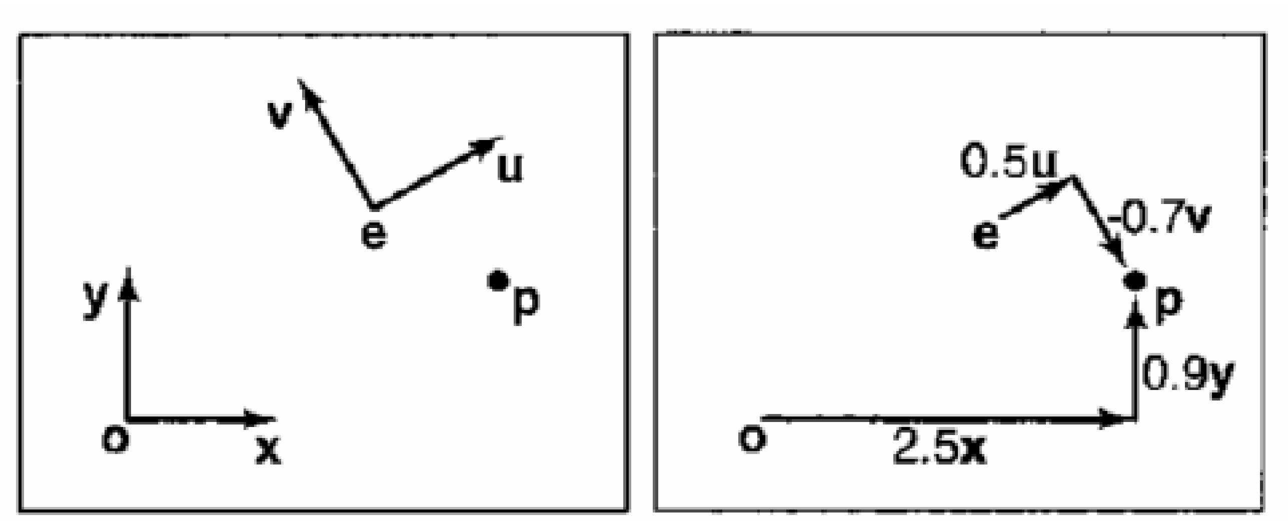
$$translation(\boldsymbol{t}) \rightarrow translation(-\boldsymbol{t})$$

$$scale(s_x, s_y, s_z) \rightarrow scale(1/s_x, 1/s_y, 1/s_z)$$

$$M_1 M_2 \ldots M_n \rightarrow M_n^{-1} \ldots M_2^{-1} M_1^{-1}$$

Coordinate Transformations

$$\boldsymbol{p} = \boldsymbol{o} + x_p \boldsymbol{x} + y_p \boldsymbol{y}$$
$$\boldsymbol{p} = \boldsymbol{e} + u_p \boldsymbol{u} + v_p \boldsymbol{v}$$

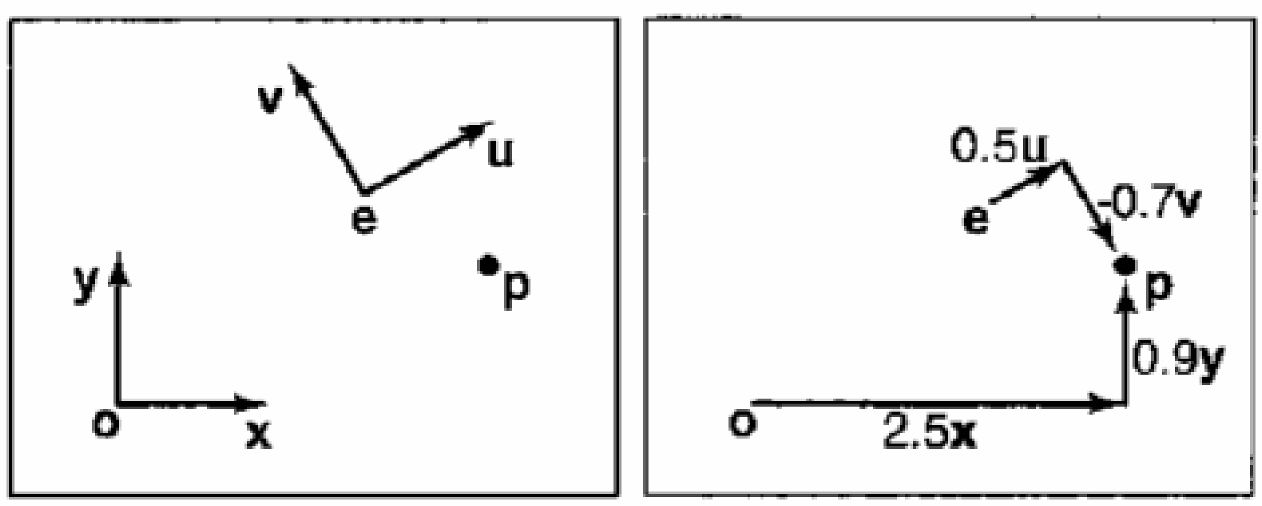


How to find (x_p, y_p) from (u_p, v_p) and vice versa?

Coordinate Transformations

$$\begin{bmatrix} x_p \\ y_p \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & x_e \\ 0 & 1 & y_e \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_u & x_v & 0 \\ y_u & y_v & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_p \\ v_p \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} u_p \\ v_p \\ 1 \end{bmatrix} = \begin{bmatrix} x_u & y_u & 0 \\ x_v & y_v & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_e \\ 0 & 1 & -y_e \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_p \\ y_p \\ 1 \end{bmatrix}$$



Recap

- 3D Transformations
- Translations
- Windowing Transforms
- Coordinate Transformation