

CMP205: Computer Graphics



Lecture 2: Transformations I

Mohamed Alaa El-Dien Aly
Computer Engineering Department
Cairo University
Fall 2013

Agenda

- 2D Transformations
- 3D Transformations
- 2D & 3D Translation

Acknowledgments: Some slides adapted from Steve Marschner and Fredo Durand.

2D Transformations

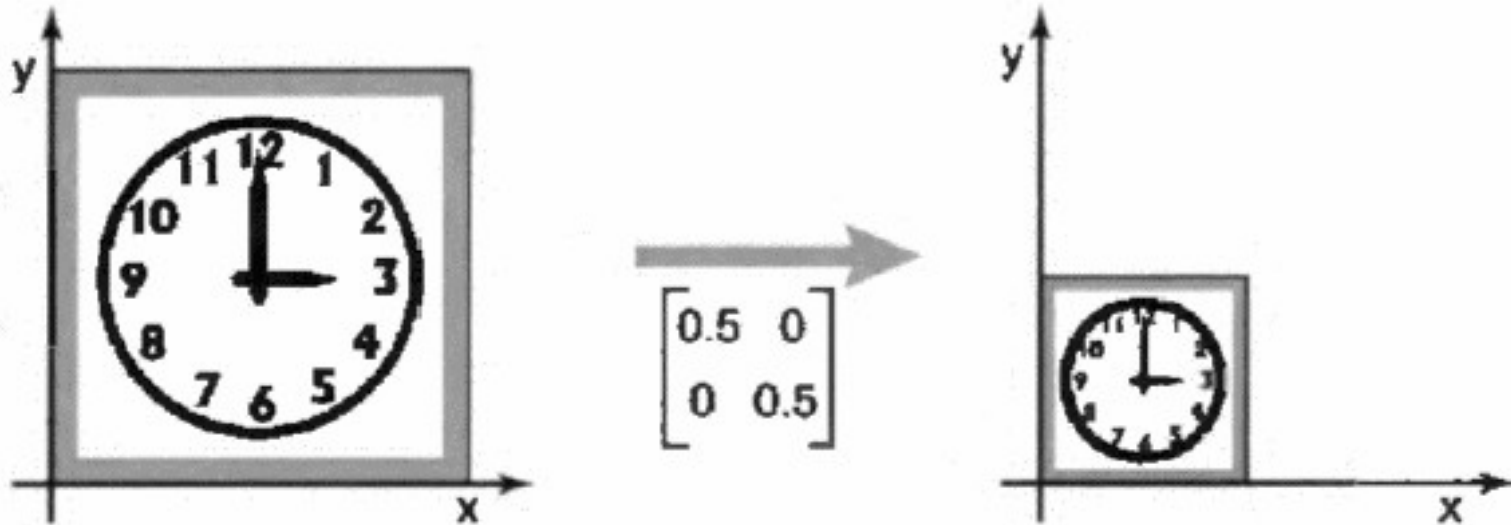
- Scale
- Shear
- Rotation
- Reflection

Look at *linear* transformations in the form:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_{11}x + a_{12}y \\ a_{21}x + a_{22}y \end{bmatrix}$$

Scale

Just “scales” all the points by multiplying them with a scale factor



Scale

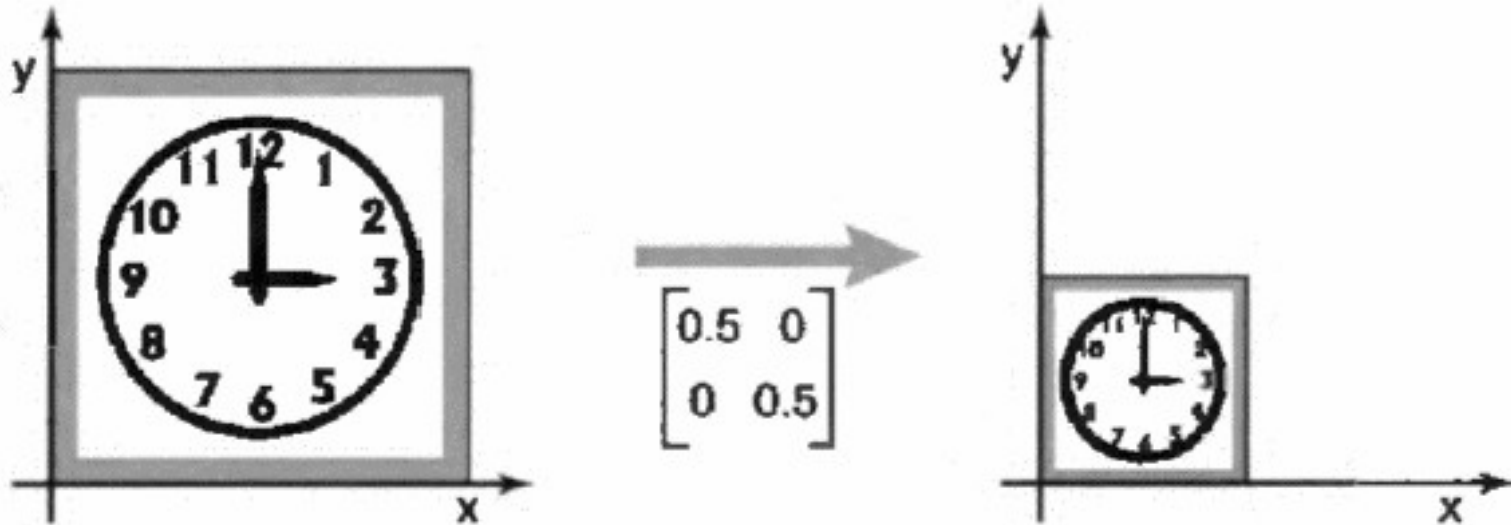
We can define it by the matrix:

$$\text{Scale}(s_x, s_y) = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} s_x x \\ s_y y \end{bmatrix}$$

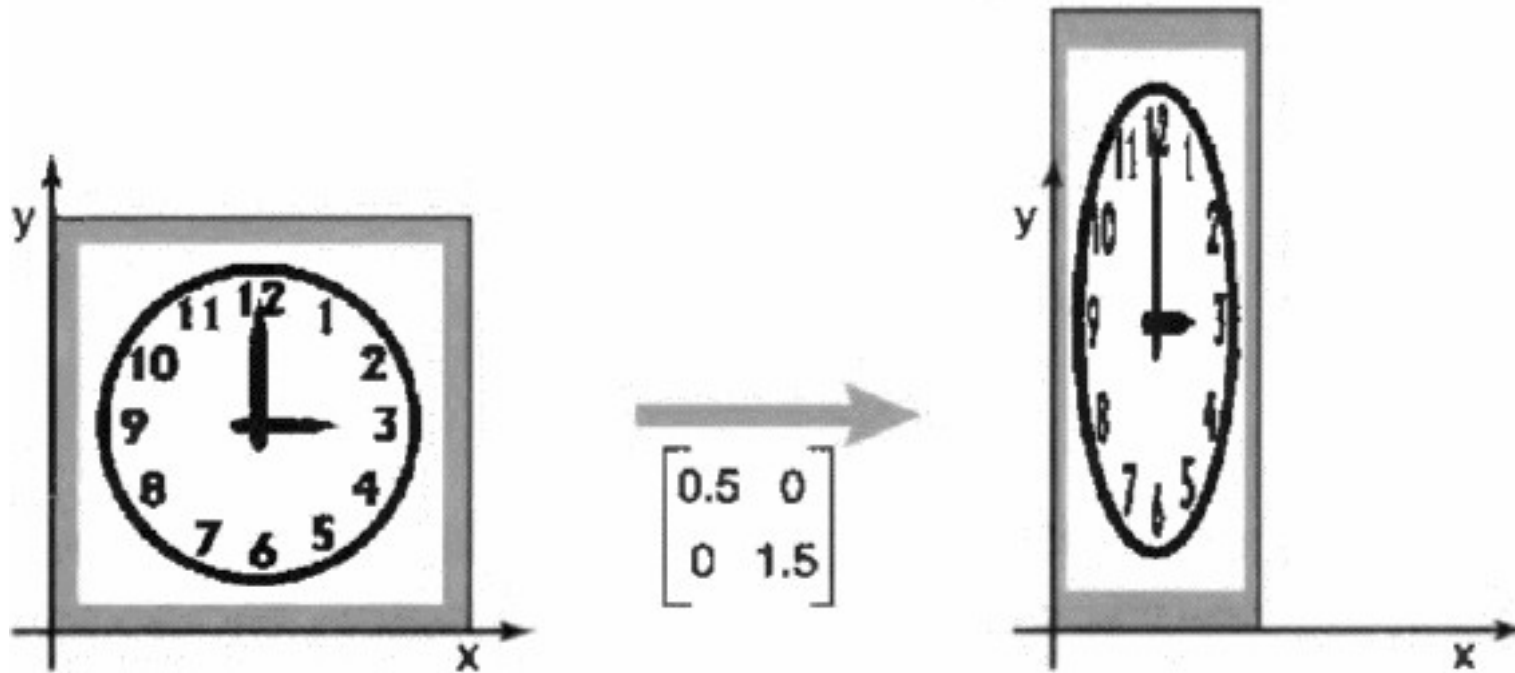
Scale

- Uniform Scale: equal factors in x and y directions



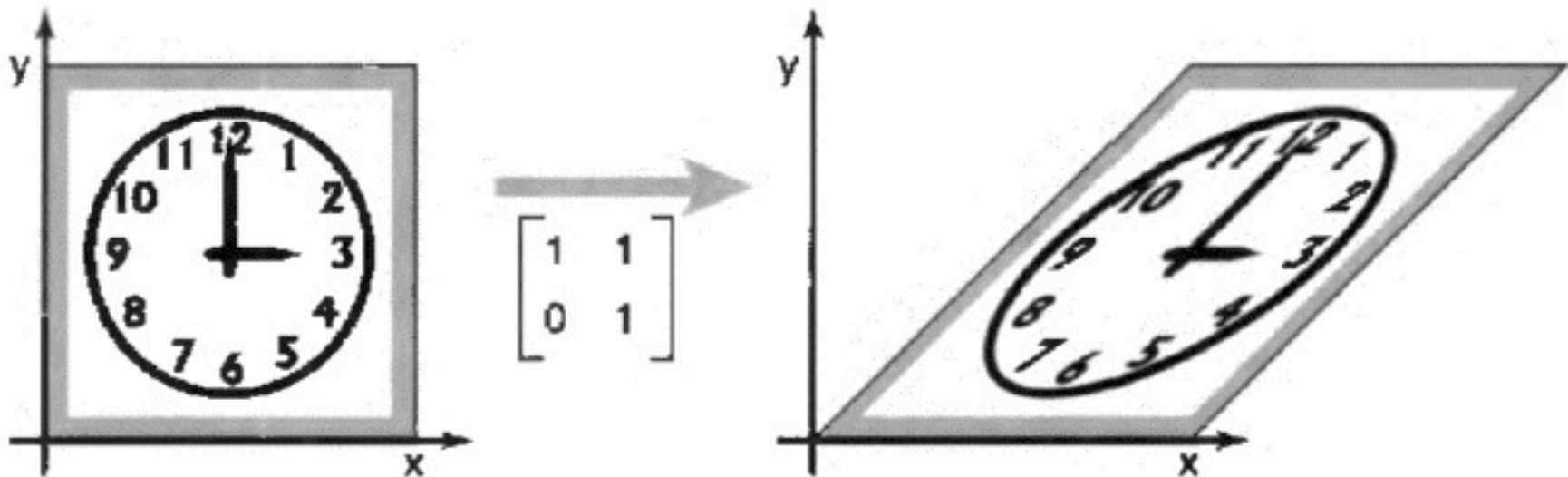
Scale

- Nonuniform Scale: different factors in x and y directions



Shearing

Shears the points by stretching along one direction



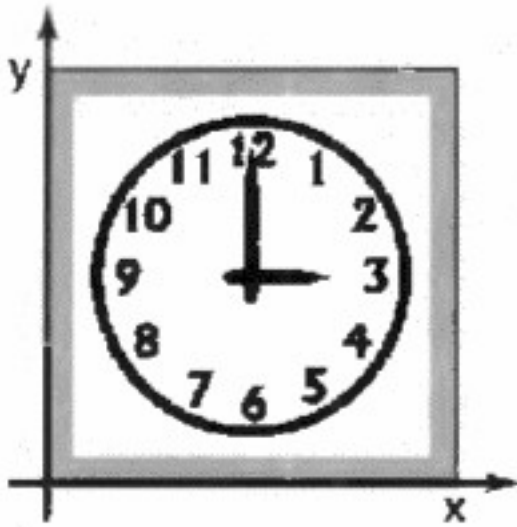
Shearing

$$\text{shear-x}(s) = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$$

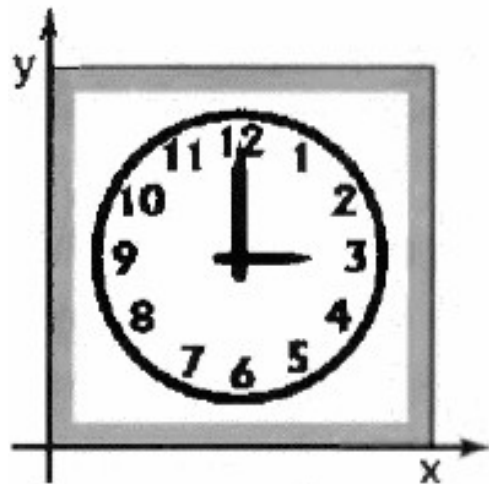
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + sy \\ y \end{bmatrix}$$

$$\text{shear-y}(s) = \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix}$$

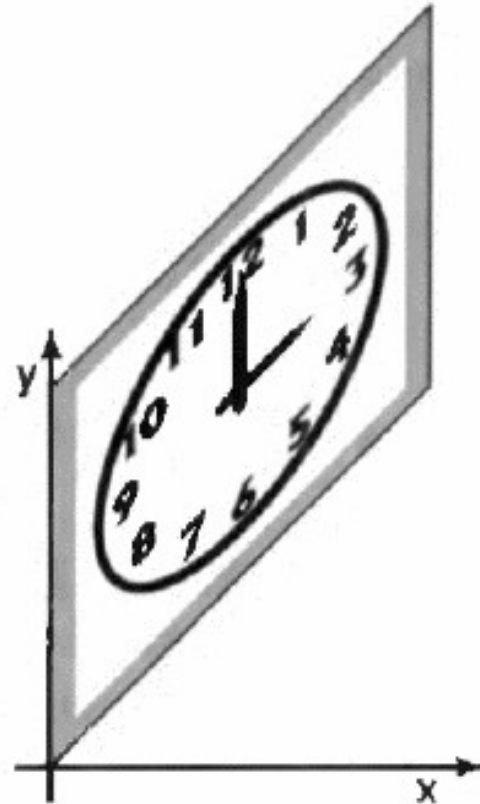
Shearing



$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

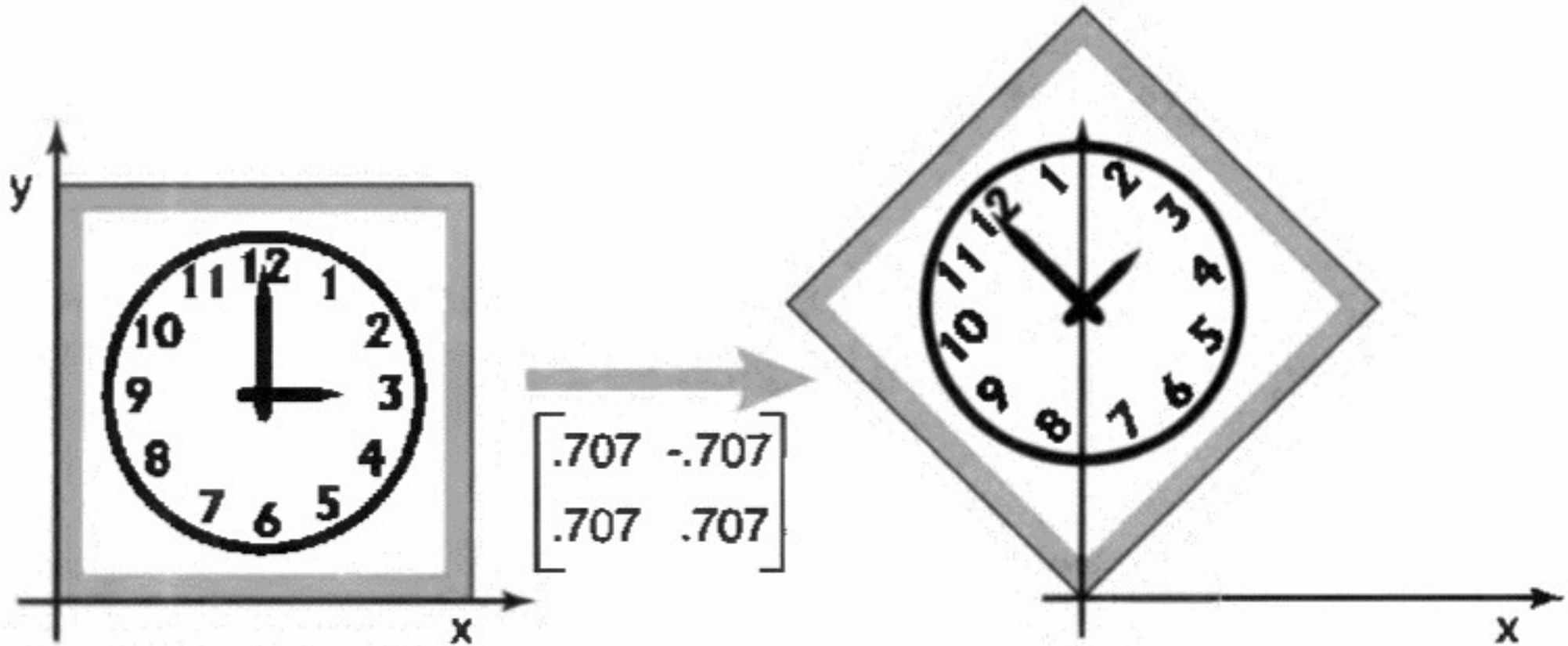


$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$



Rotation

Rotate all the points around the origin



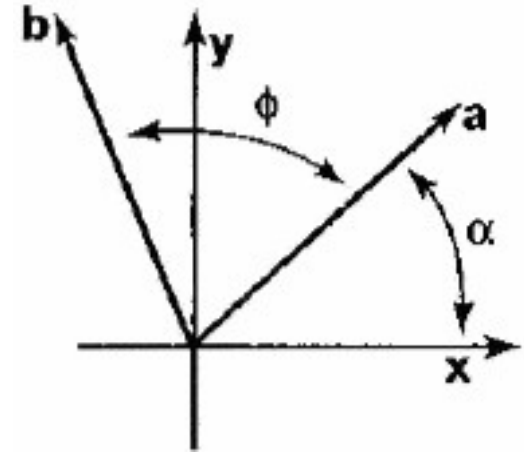
Rotation

$$\begin{aligned}x_a &= r \cos \alpha & x_b &= r \cos (\alpha + \varphi) \\y_a &= r \sin \alpha & y_b &= r \sin (\alpha + \varphi)\end{aligned}$$

and we know that:

$$\begin{aligned}\cos (\alpha + \varphi) &= \cos \alpha \cos \varphi - \sin \alpha \sin \varphi \\ \sin (\alpha + \varphi) &= \cos \alpha \sin \varphi + \sin \alpha \cos \varphi\end{aligned}$$

$$\begin{aligned}x_b &= x_a \cos \varphi - y_a \sin \varphi \\ y_b &= x_a \sin \varphi + y_a \cos \varphi\end{aligned}$$



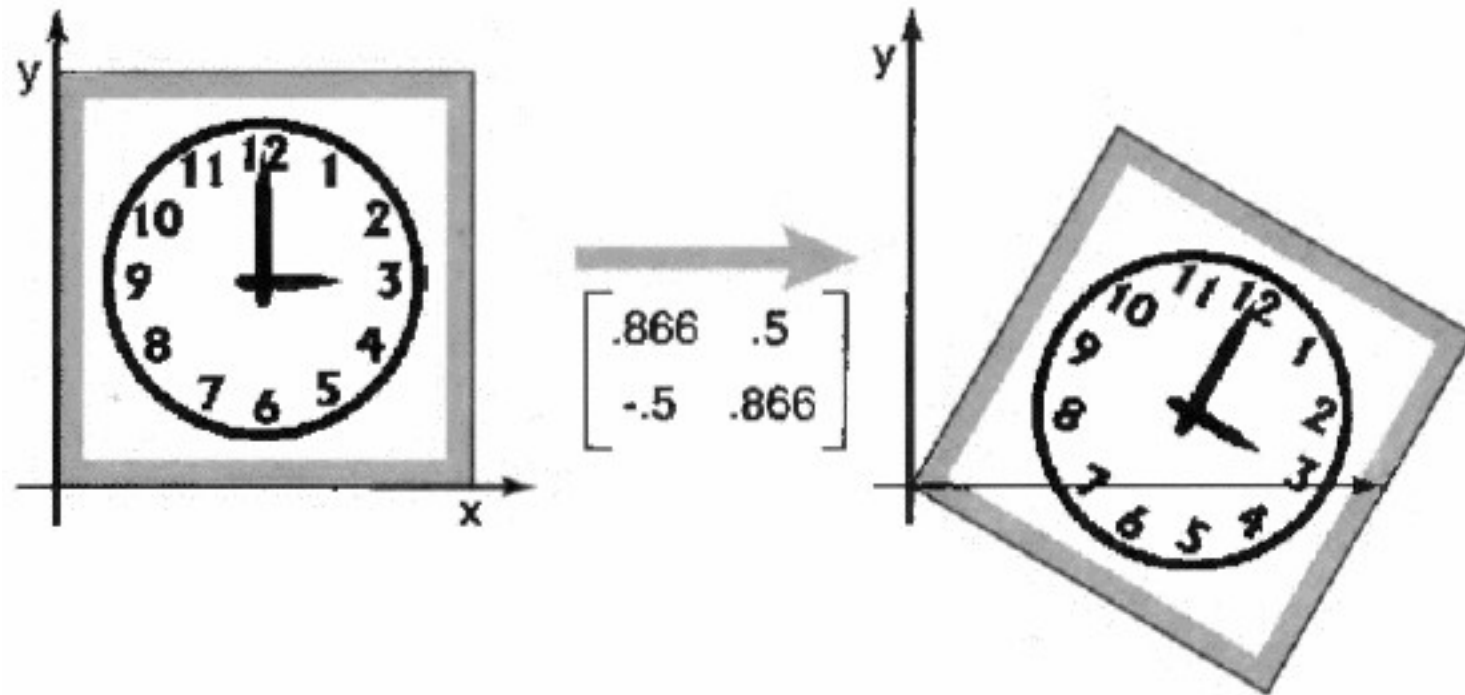
Rotation

$$\text{rotate}(\varphi) = \begin{bmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{bmatrix}$$

What's the important property of this matrix?

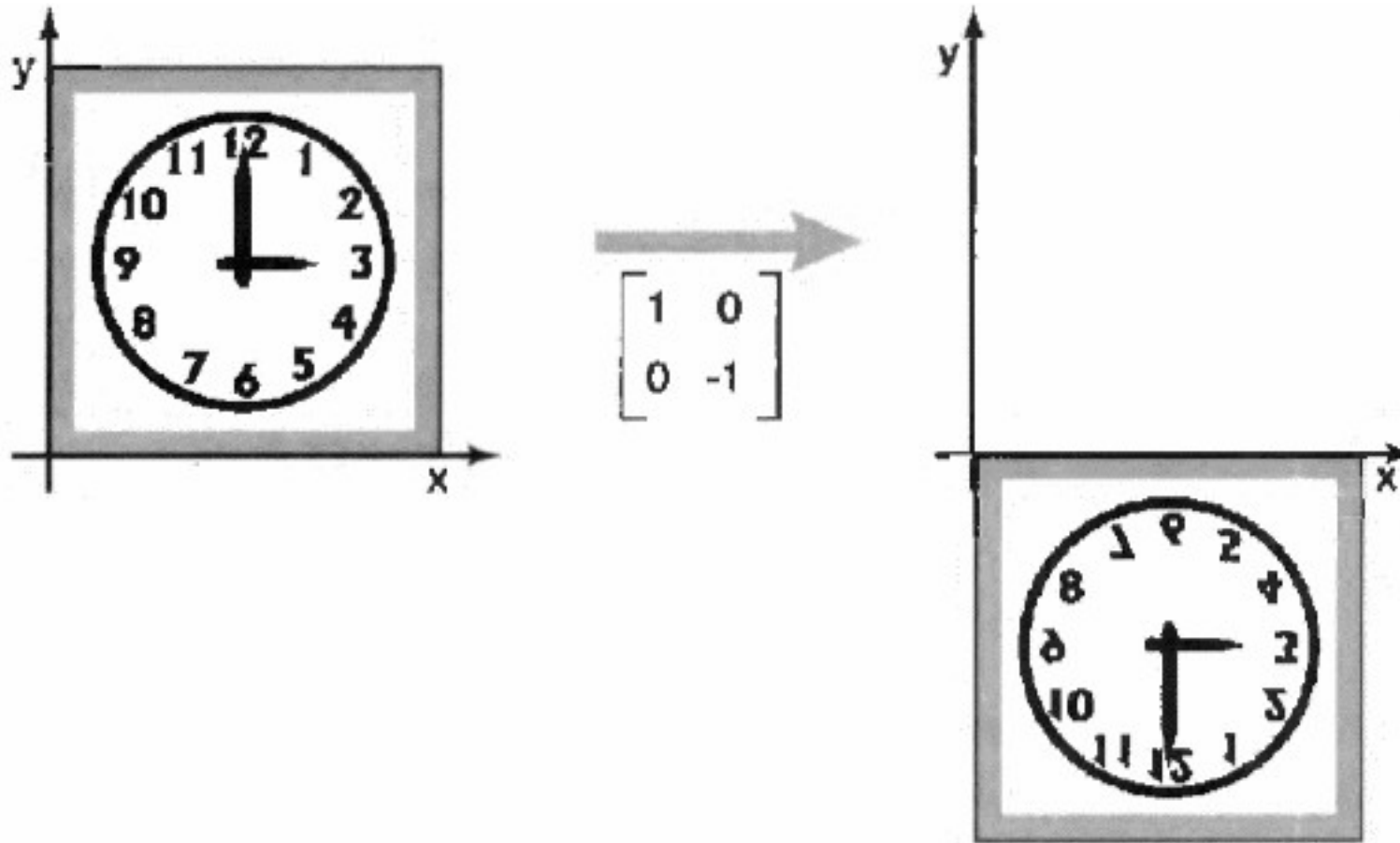
It's orthonormal i.e. $R R^T = I$

Rotation



Reflection

Reflects points around some axis



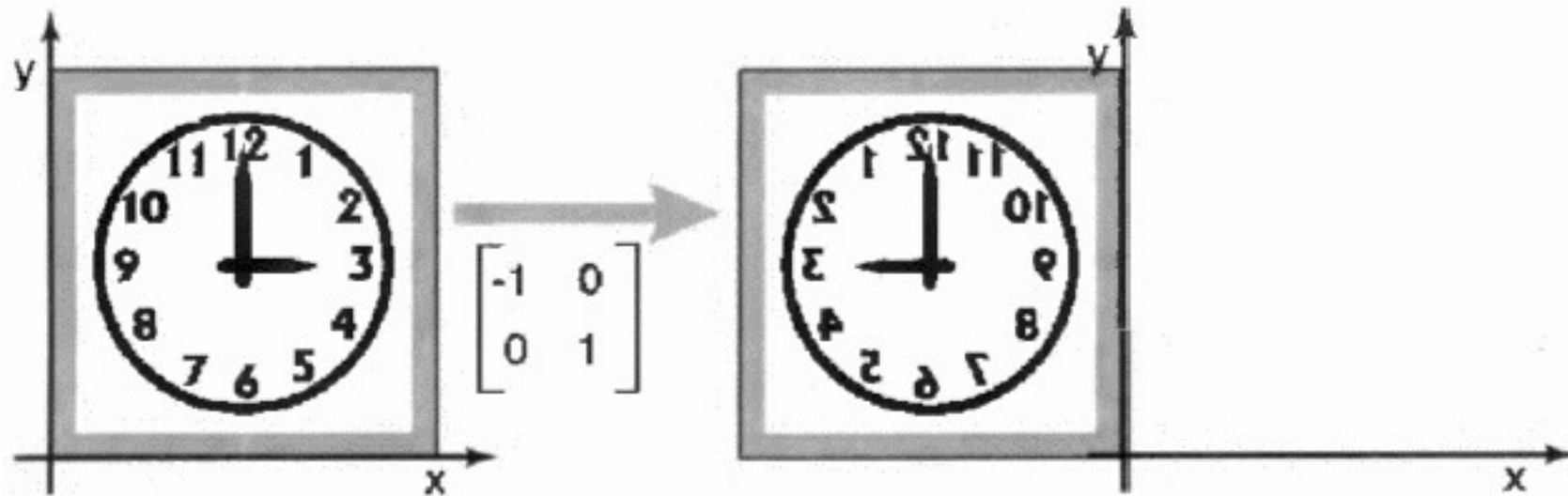
Reflection

$$\text{reflect-x} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$$

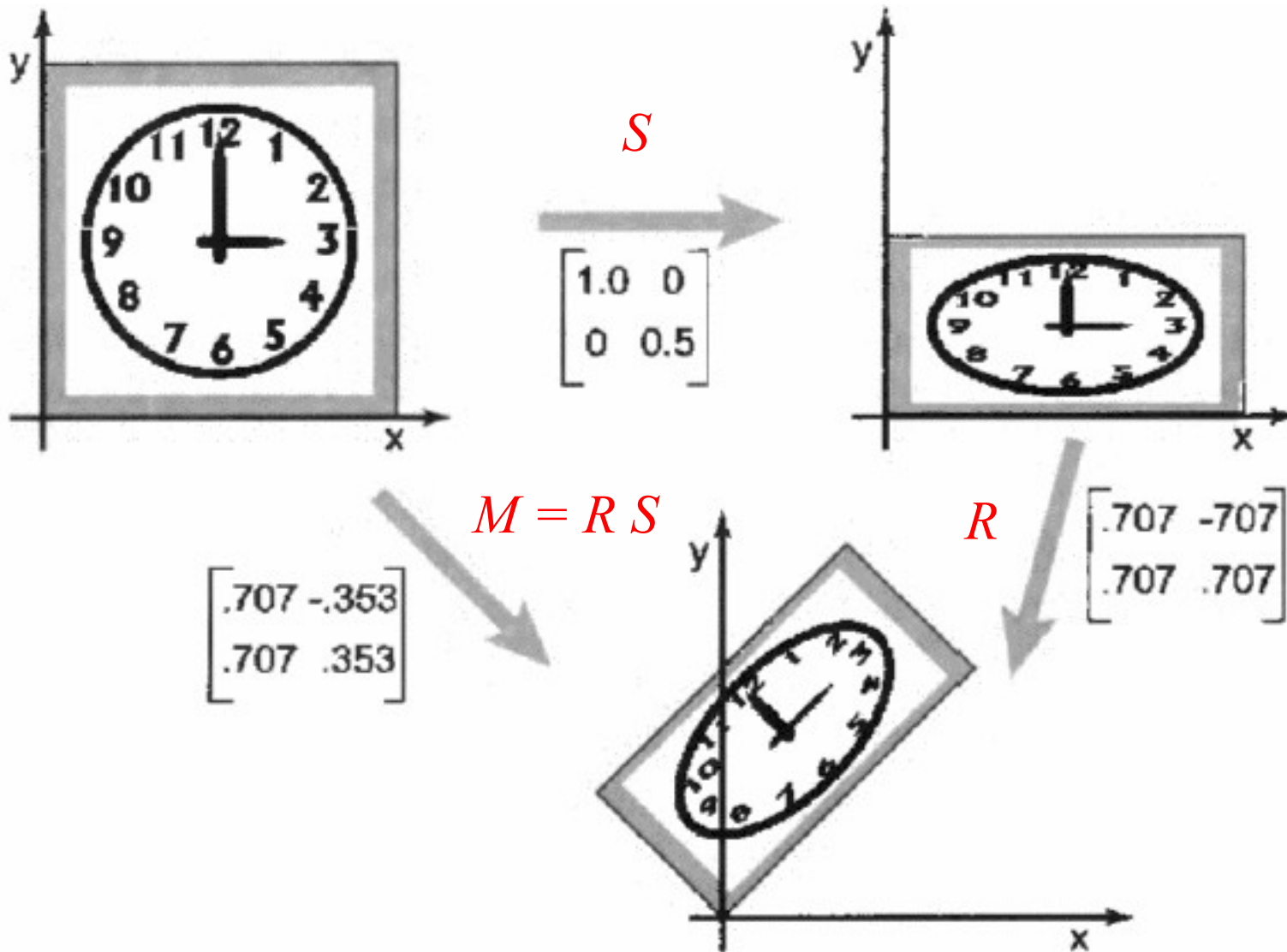
$$\text{reflect-y} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Reflection



Composition of 2D Transforms

Two (or more) transformation matrices can be combined in one matrix



$$v_2 = S v_1$$

$$v_3 = R v_2$$

$$\downarrow$$

$$v_3 = (RS) v_1$$

$$v_3 = M v_1$$

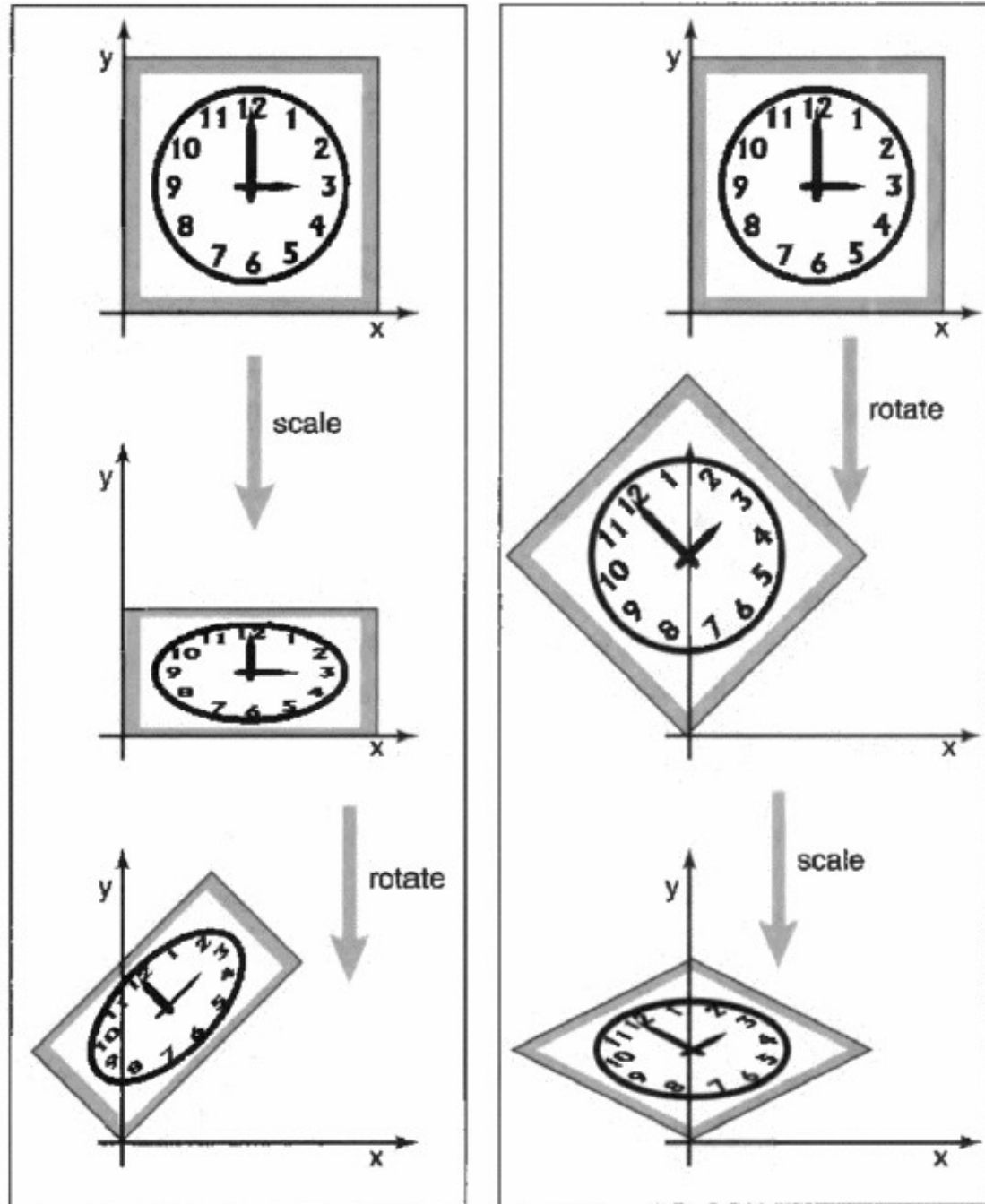
where:

$$M = RS$$

Composition of 2D Transforms

Beware that the order of the transformations matters!

$$RS \neq SR$$



3D Scaling

Here we have scaling in three dimensions instead of two!

$$\text{scale}(s_x, s_y, s_z) = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{bmatrix}$$

3D Rotation

Here we have three possible *standard* axes to rotate around:

$$\text{rotate-z}(\varphi) = \begin{bmatrix} \cos \varphi & -\sin \varphi & 0 \\ \sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{rotate-x}(\varphi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \varphi & -\sin \varphi \\ 0 & \sin \varphi & \cos \varphi \end{bmatrix}$$

$$\text{rotate-y}(\varphi) = \begin{bmatrix} \cos \varphi & 0 & \sin \varphi \\ 0 & 1 & 0 \\ -\sin \varphi & 0 & \cos \varphi \end{bmatrix}$$

3D Shear

Instead of just *one* factor for y , we have *two* factors for y and z :

$$\text{shear-x}(d_y, d_z) = \begin{bmatrix} 1 & d_y & d_z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2D Translations

2D Transformations

$$p' = M_{2 \times 2} p$$

$$x' = m_{11}x + m_{12}y$$

$$y' = m_{21}x + m_{22}y$$

Translation

$$x' = x + t_x$$

$$y' = y + t_y$$

How can we represent *Translation* as Matrix Multiplication?

2D Translations

3D Shear in XY plane

$$\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + t_x z \\ y + t_y z \\ z \end{bmatrix}$$

Translation

$$\begin{aligned} x' &= x + t_x z \\ y' &= y + t_y z \end{aligned}$$

Solution: add $z=1$ to 2D points

$$\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix}$$

What about vectors?

$$z = 0 !$$

Homogeneous Coordinates

Convert 2D points into 3D points

$$v = \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \tilde{v} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

2D Transformations with 3x3 matrix

$$\tilde{v}' = M \tilde{v}$$

2D Transformations

Translation

$$\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

Rotation

$$\begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$v = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Shear

$$\begin{bmatrix} 1 & s & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Scaling

$$\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Can represent any combination by a 3x3 matrix

2D Transformations

Rotation/Scale/Shear + Translation

$$\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} R_{2 \times 2} & t_{2 \times 1} \\ 0^T & 1 \end{bmatrix}$$

Rotation part + translation part

3D Translations

Homogeneous Coordinates

$$\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x + t_x \\ y + t_y \\ z + t_z \\ 1 \end{bmatrix}$$

3D Transformations

Translation

$$\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$v = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Can represent any combination by a 4x4 matrix

3D Transformations

Rotation/Scale/Shear + Translation

$$\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & t_x \\ a_{21} & a_{22} & a_{23} & t_y \\ a_{31} & a_{32} & a_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} R_{3 \times 3} & t_{3 \times 1} \\ 0^T & 1 \end{bmatrix}$$

Rotation part + translation part

Transformation Inverse

$$M \rightarrow M^{-1}$$

$$\text{Rotation } R \rightarrow R^T$$

$$\text{translation}(\mathbf{t}) \rightarrow \text{translation}(-\mathbf{t})$$

$$\text{scale}(s_x, s_y, s_z) \rightarrow \text{scale}(1/s_x, 1/s_y, 1/s_z)$$

$$M_1 M_2 \dots M_n \rightarrow M_n^{-1} \dots M_2^{-1} M_1^{-1}$$

Recap

- 2D Transformations
- 3D Transformations
- 2D & 3D Translation