CMP302: Algorithms

Lecture 02: Merge Sort and Asymptotic Analysis

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Fall 2013

Agenda

- Merge Sort
- Recurrences
- Asymptotic Notation

Acknowledgment
A lot of slides adapted from the slides of Erik Demaine and Charles Leiserson

Merge Sort

- Divide-and-Conquer
  - *Divide* the problem into a number of sub-problems
  - *Conquer* the smaller problems
  - *Combine* the results of the sub-problems into a solution for the big problem

```
MERGE-SORT(A, p, r)
1  if p < r
2     then q ← ⌊(p + r)/2⌋
3          MERGE-SORT(A, p, q)
4          MERGE-SORT(A, q + 1, r)
5          MERGE(A, p, q, r)
```

Merge Sort

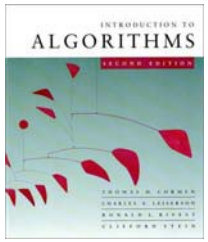
- Divide-and-Conquer
  - *Divide* the problem into a number of sub-problems
  - *Conquer* the smaller problems
  - *Combine* the results of the sub-problems into a solution for the big problem

```
void merge_sort(vector<int>& A, int p, int r) {
  if (p >= r) return;

  int q = (p + r) / 2;

  merge_sort(A, p, q);
  merge_sort(A, q+1, r);

  merge(A, p, q, r);
}
```

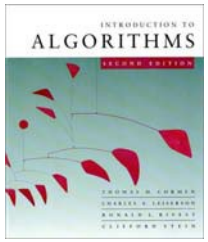


Merge sort

MERGE-SORT $A[1 \ldots n]$

1. If $n = 1$, done.
2. Recursively sort $A[1 \ldots \lfloor n/2 \rfloor]$ and $A[\lfloor n/2 \rfloor + 1 \ldots n]$.
3. "***Merge***" the 2 sorted lists.

Key subroutine: MERGE



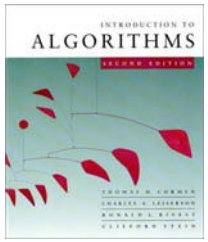
Merging two sorted arrays

20 12

13 11

7 9

2 1



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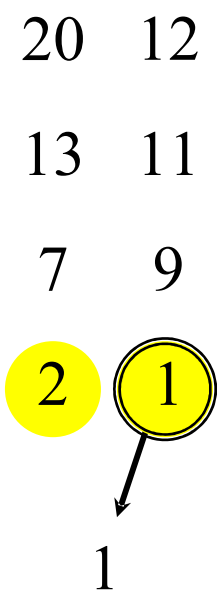

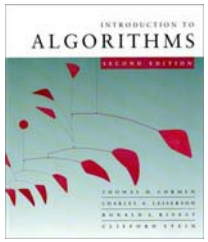

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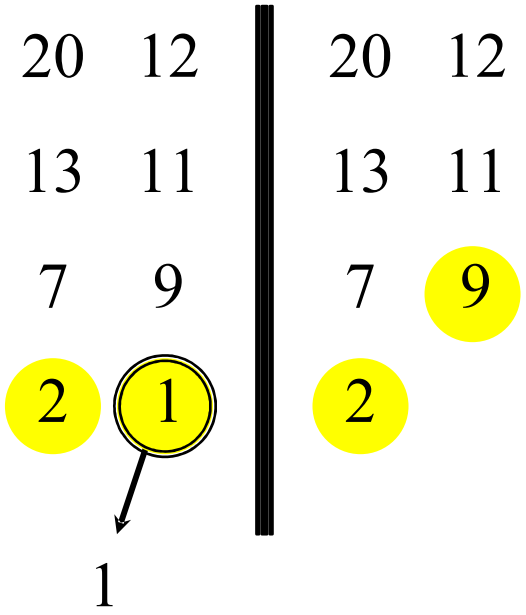

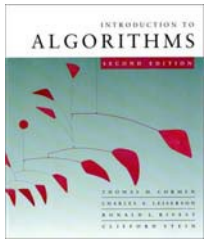

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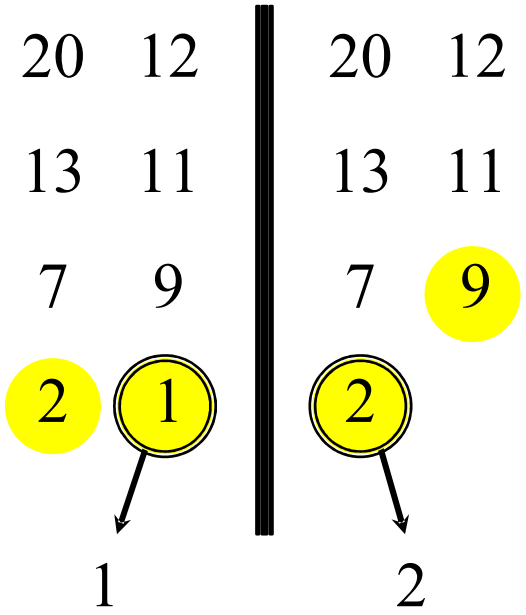

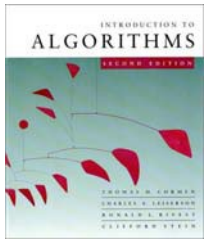

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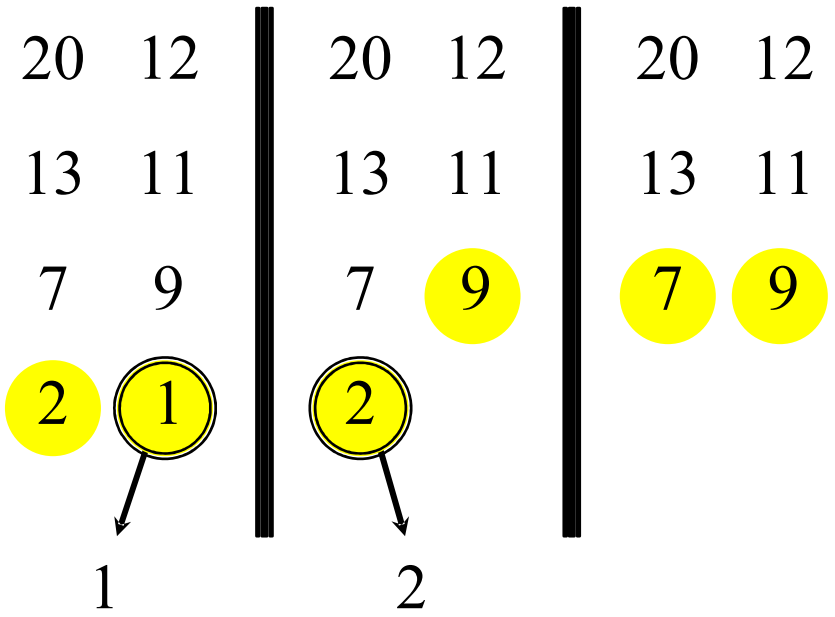

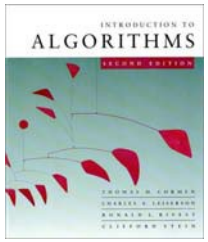

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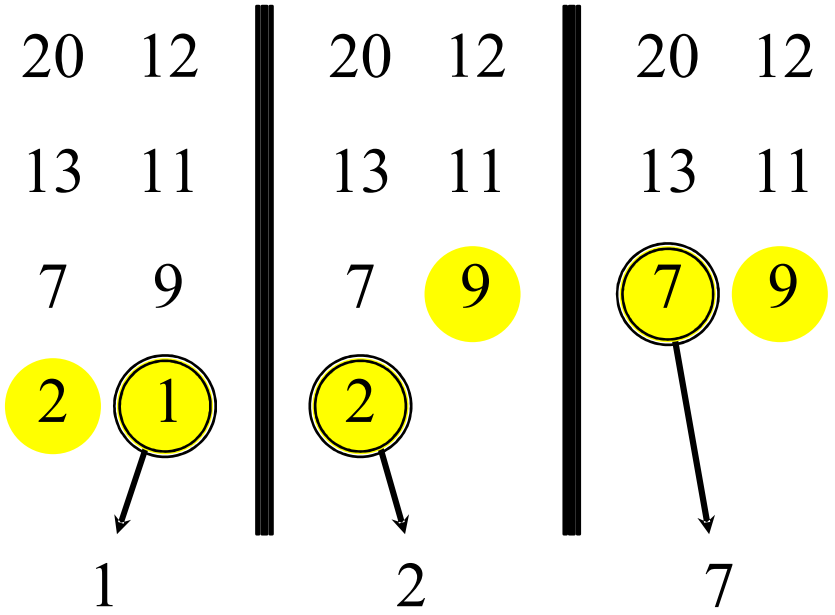

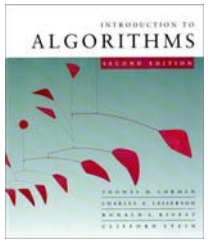

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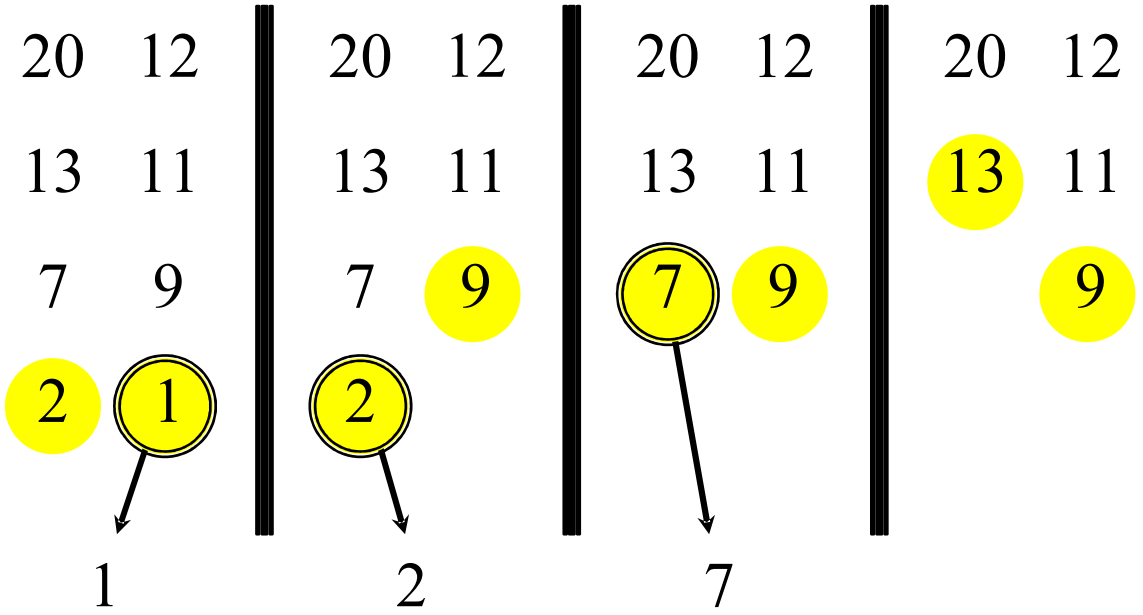

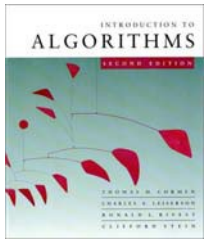

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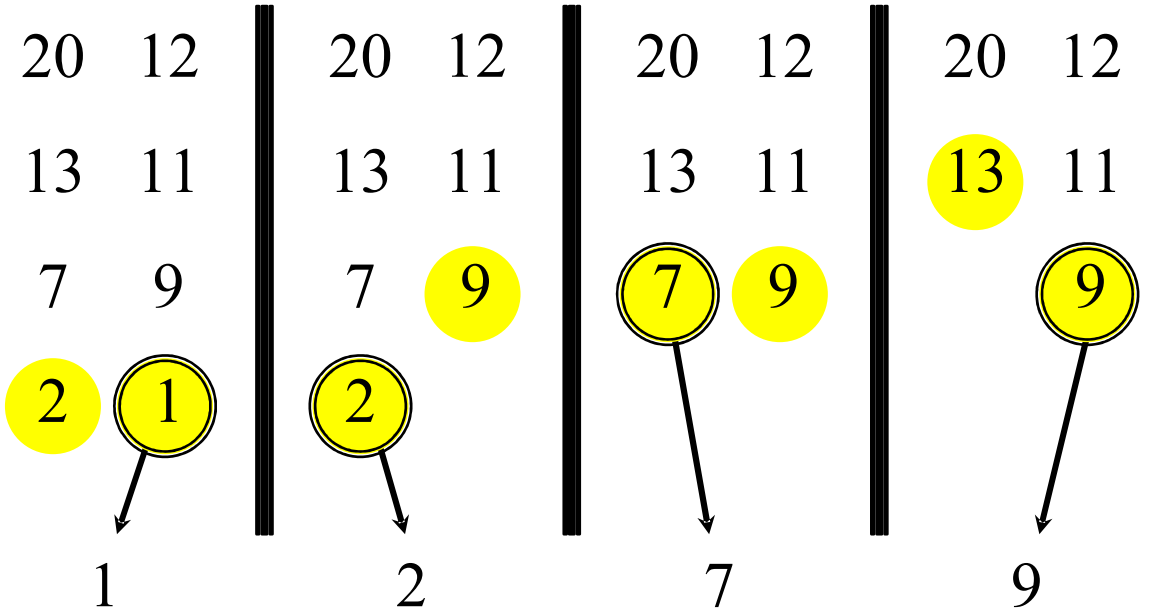

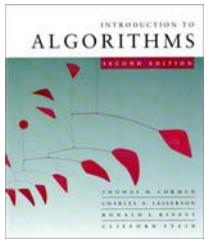

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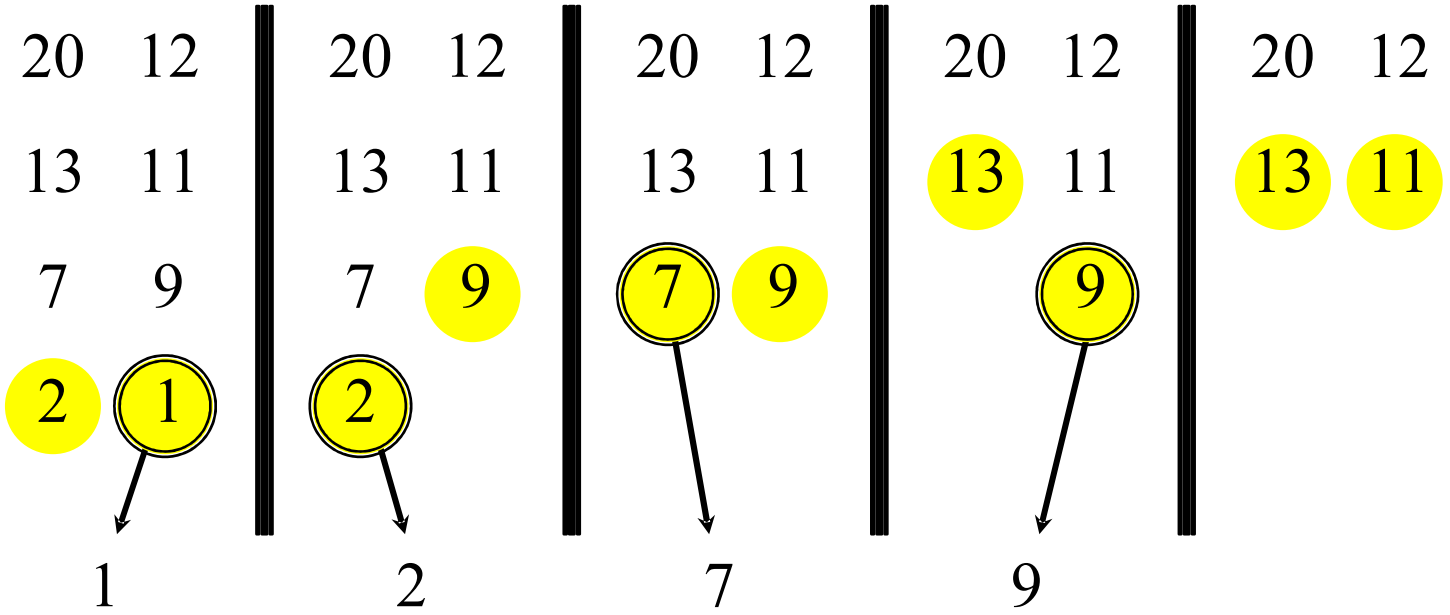

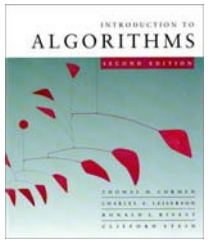

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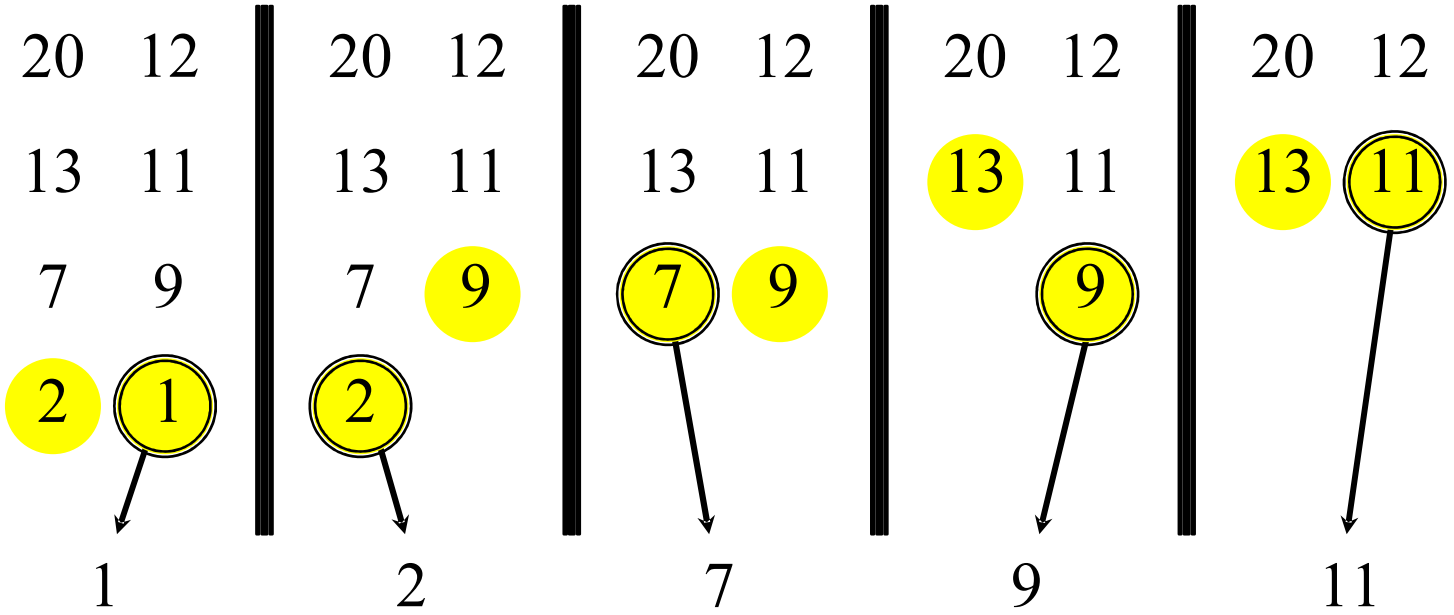

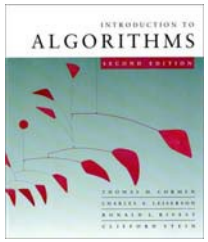

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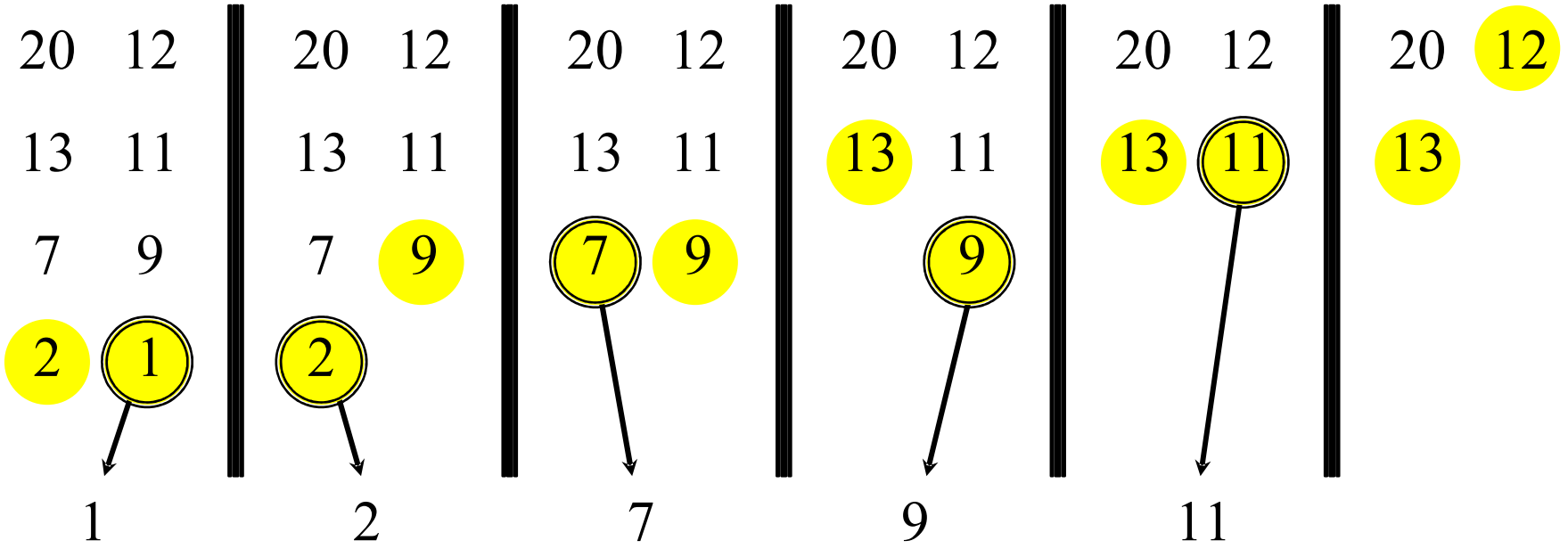

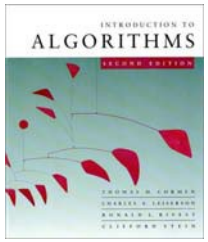

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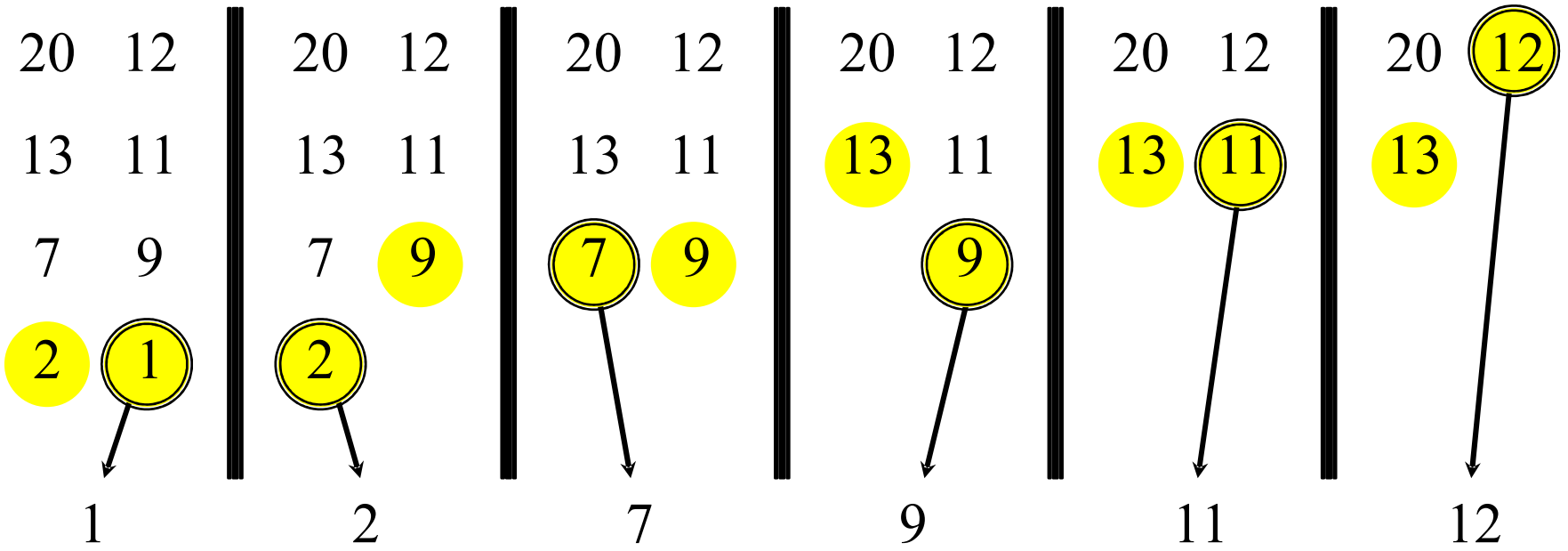

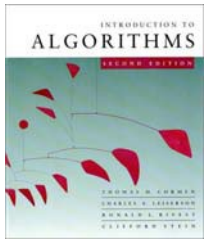

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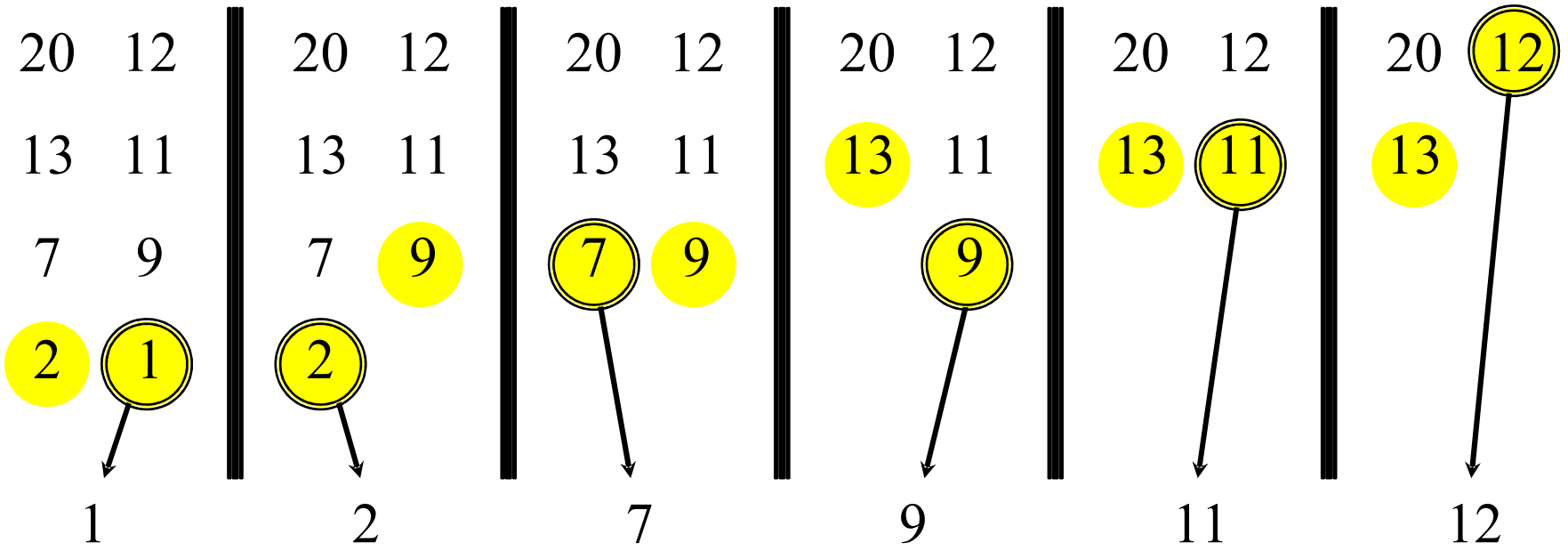


Time = $\Theta(n)$ to merge a total of n elements (linear time).

Merge

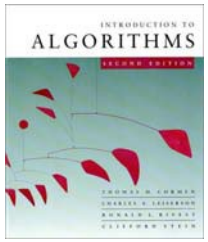
```
MERGE(A, p, q, r)
 1  n_1 = q − p + 1
 2  n_2 = r − q
 3  let L[1..n_1 + 1] and R[1..n_2 + 1] be new arrays
 4  for i = 1 to n_1
 5      L[i] = A[p + i − 1]
 6  for j = 1 to n_2
 7      R[j] = A[q + j]
 8  L[n_1 + 1] = ∞
 9  R[n_2 + 1] = ∞
10  i = 1
11  j = 1
12  for k = p to r
13      if L[i] ≤ R[j]
14          A[k] = L[i]
15          i = i + 1
16      else A[k] = R[j]
17          j = j + 1
```

compute sizes

copy arrays to be merged

adds "sentinel"

merge



Analyzing merge sort

$T(n)$	**MERGE-SORT** $A[1 \,..\, n]$
$\Theta(1)$	1. If $n = 1$, done.
$2T(n/2)$	2. Recursively sort $A[\,1 \,..\, \lfloor n/2 \rfloor\,]$ and $A[\,\lfloor n/2 \rfloor + 1 \,..\, n\,]$.
$\Theta(n)$	***3. "Merge"*** the 2 sorted lists

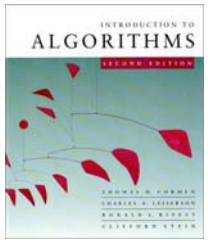
Abuse

Sloppiness: Should be $T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil)$, but it turns out not to matter asymptotically.

Recurrence for Merge Sort

$$T(n) = \begin{cases} \Theta(1) \text{ if } n = 1; \\ 2T(n/2) + \Theta(n) \text{ if } n > 1. \end{cases}$$

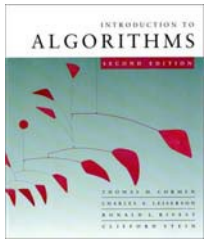
- Next we need to solve this recurrence relation i.e. find *T(n)* as a function of *n* without *T(n/2)*



Recurrence for merge sort

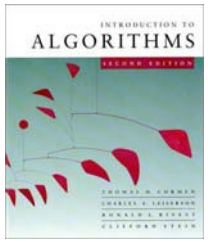
$$T(n) = \begin{cases} \Theta(1) \text{ if } n = 1; \\ 2T(n/2) + \Theta(n) \text{ if } n > 1. \end{cases}$$

- We shall usually omit stating the base case when $T(n) = \Theta(1)$ for sufficiently small n, but only when it has no effect on the asymptotic solution to the recurrence.
- CLRS and Lecture 2 provide several ways to find a good upper bound on $T(n)$.



Recursion tree

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.



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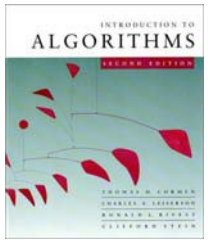
$T(n)$



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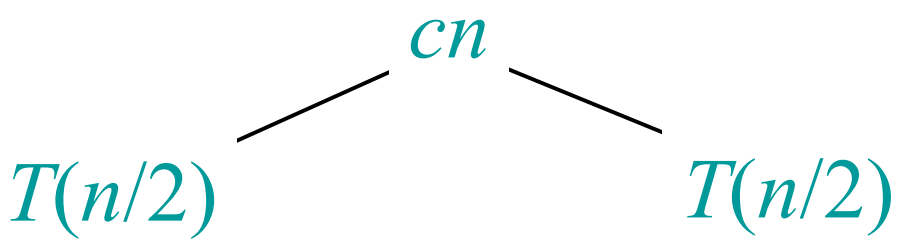

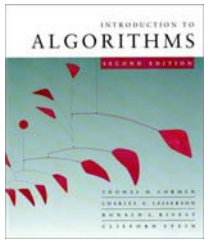

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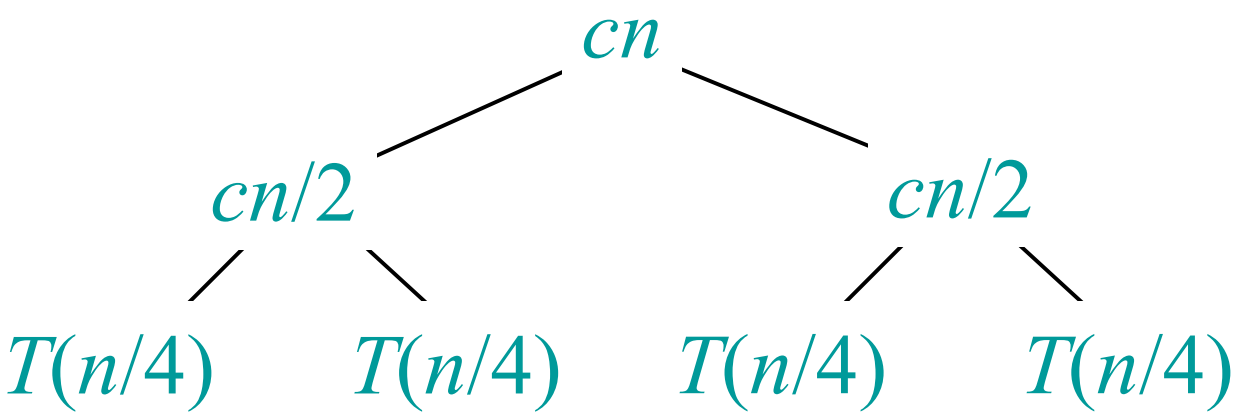

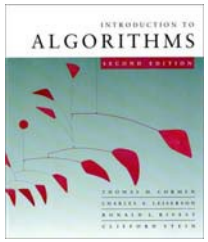

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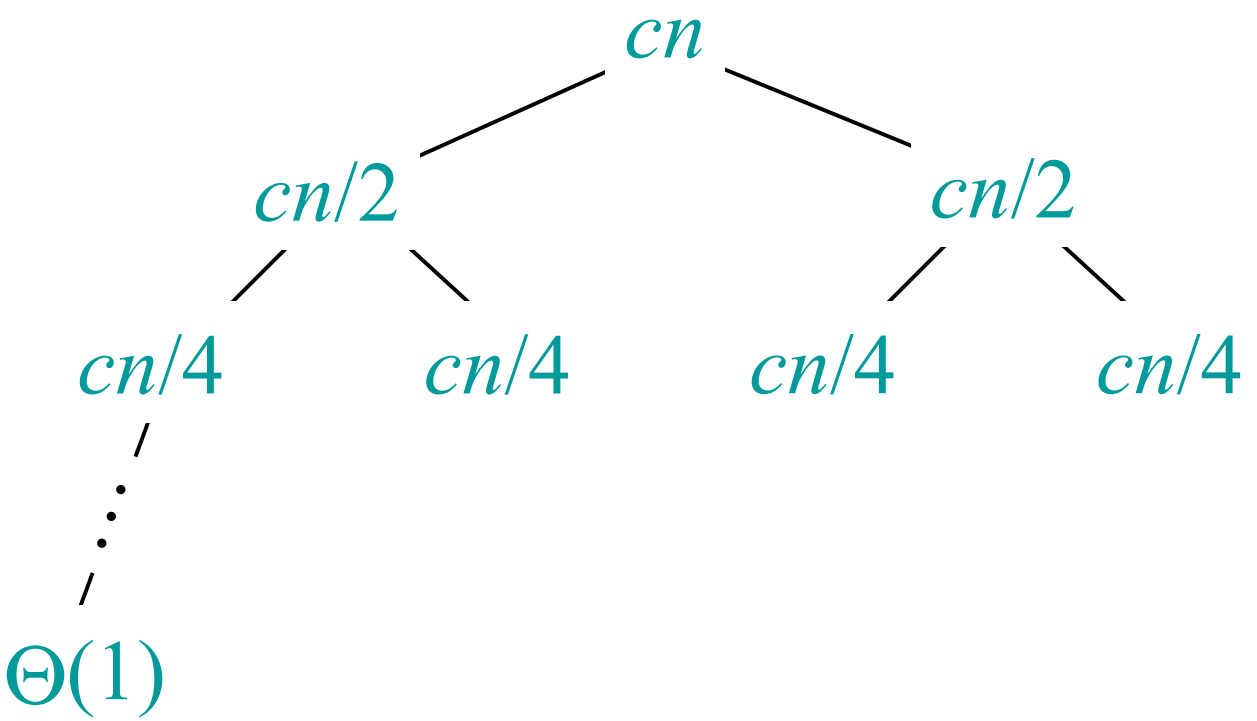

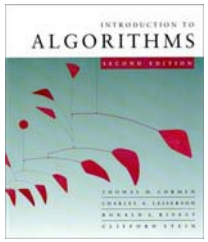

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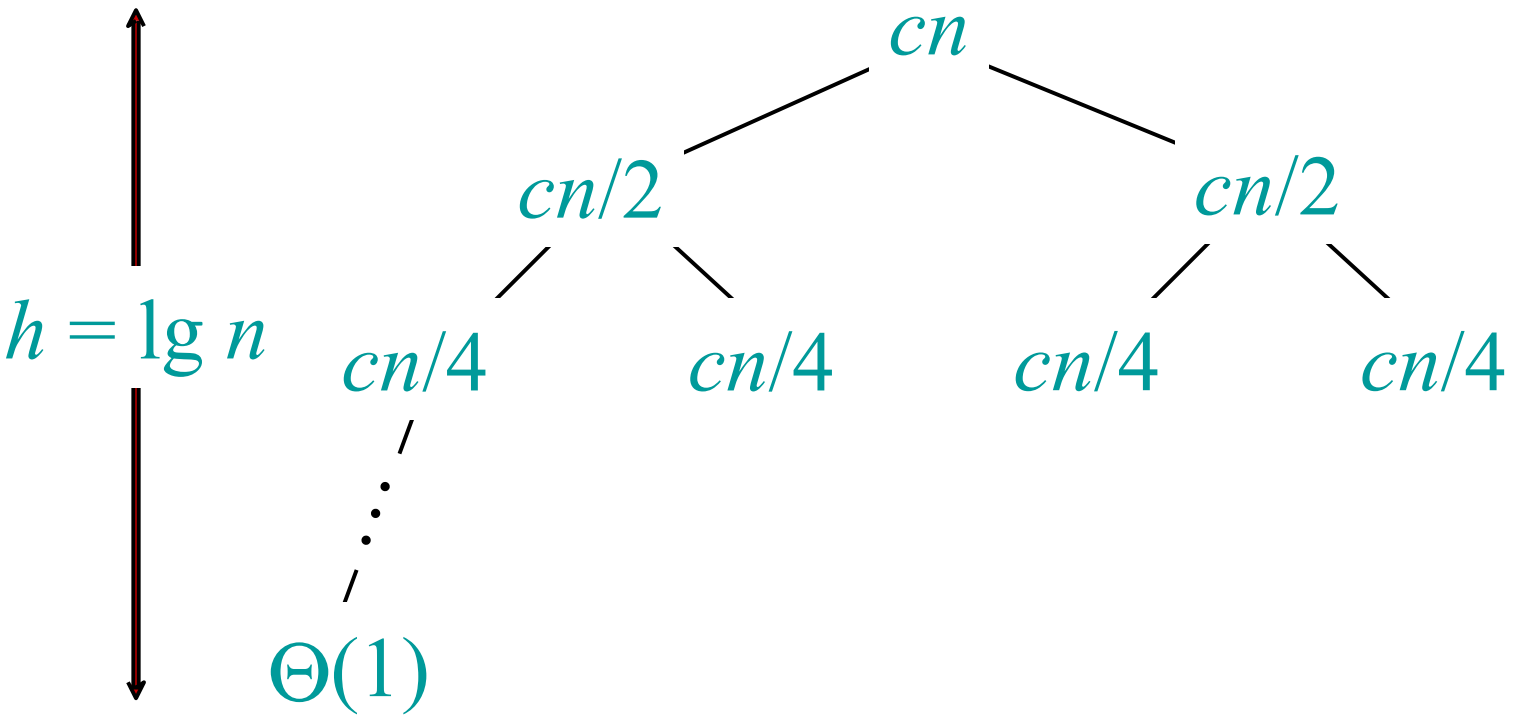

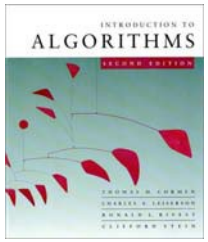

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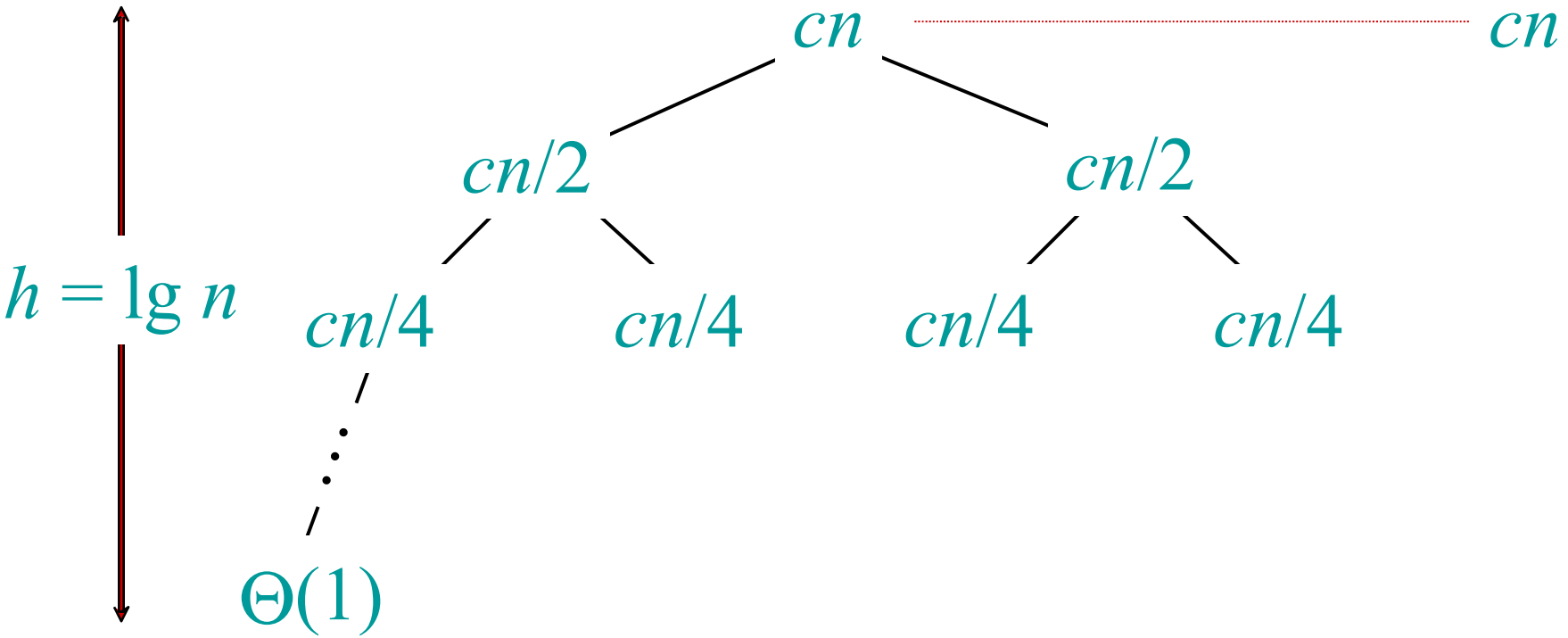

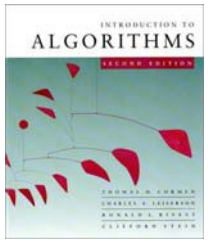

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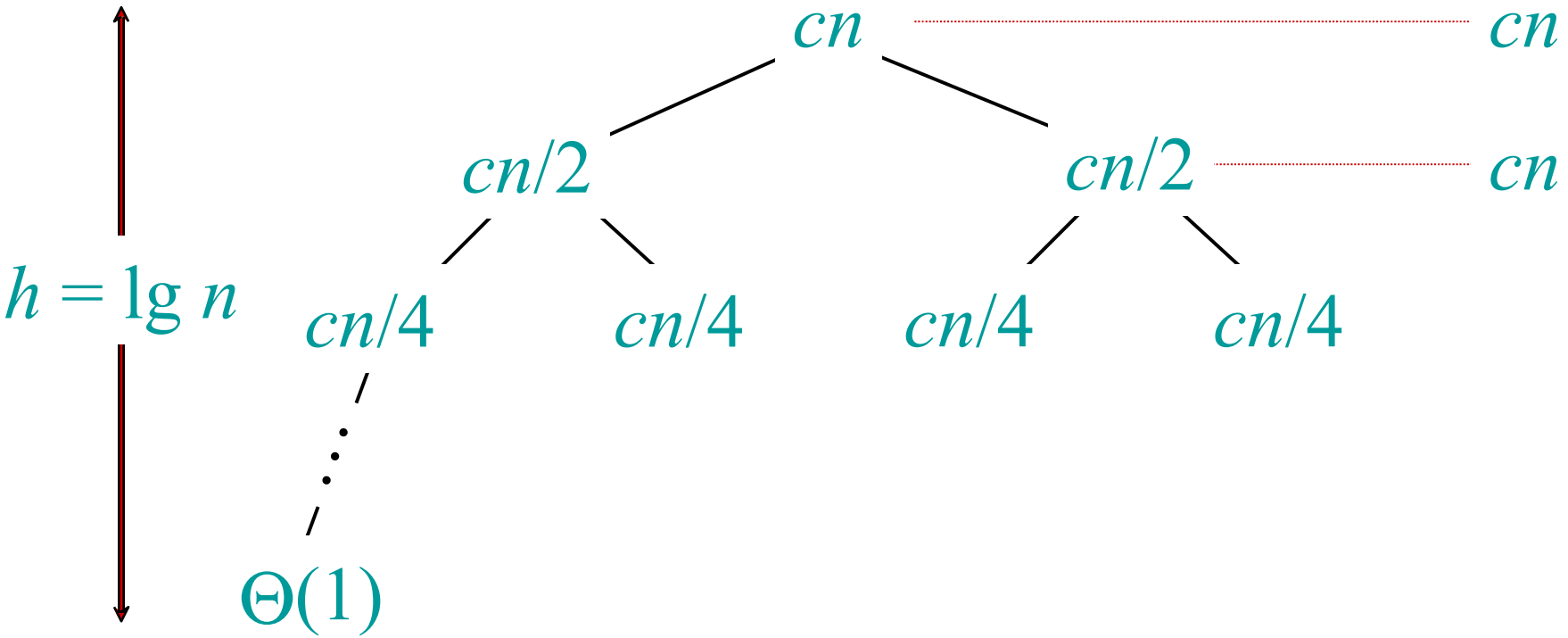

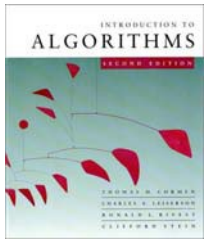

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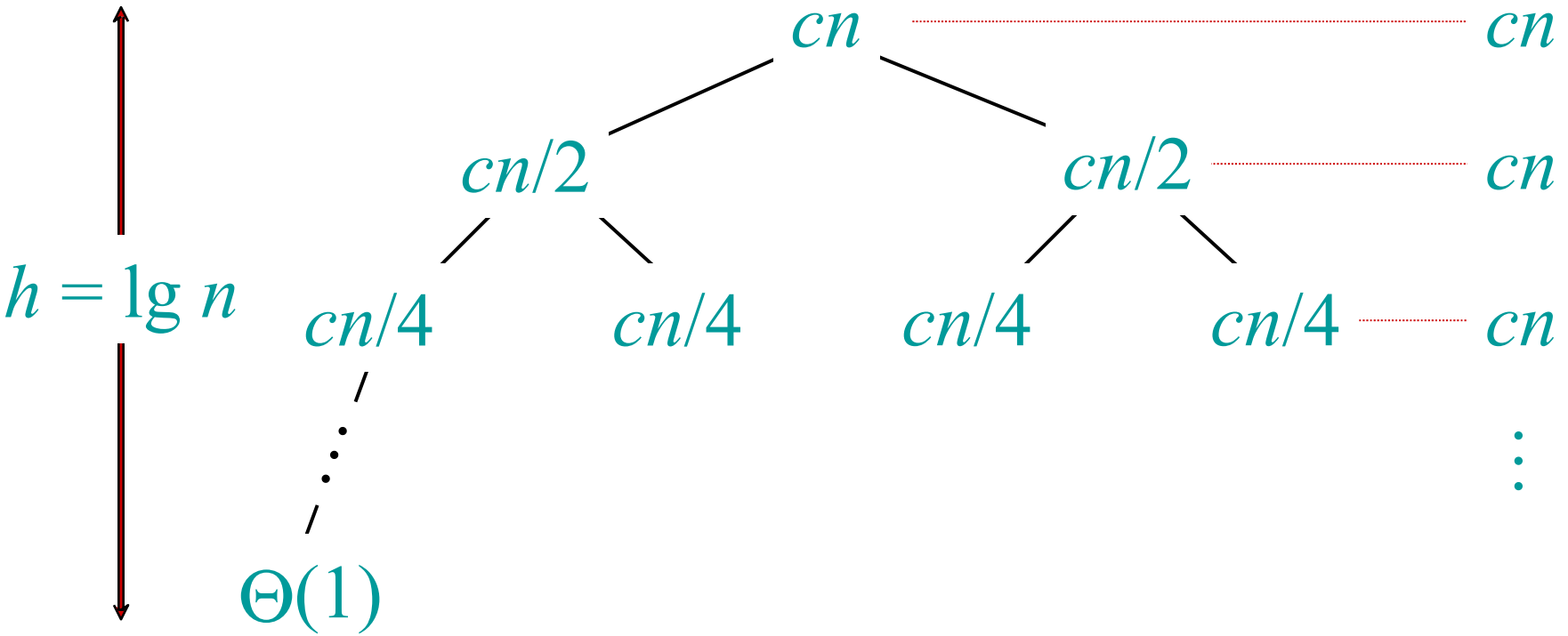

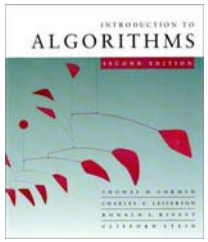

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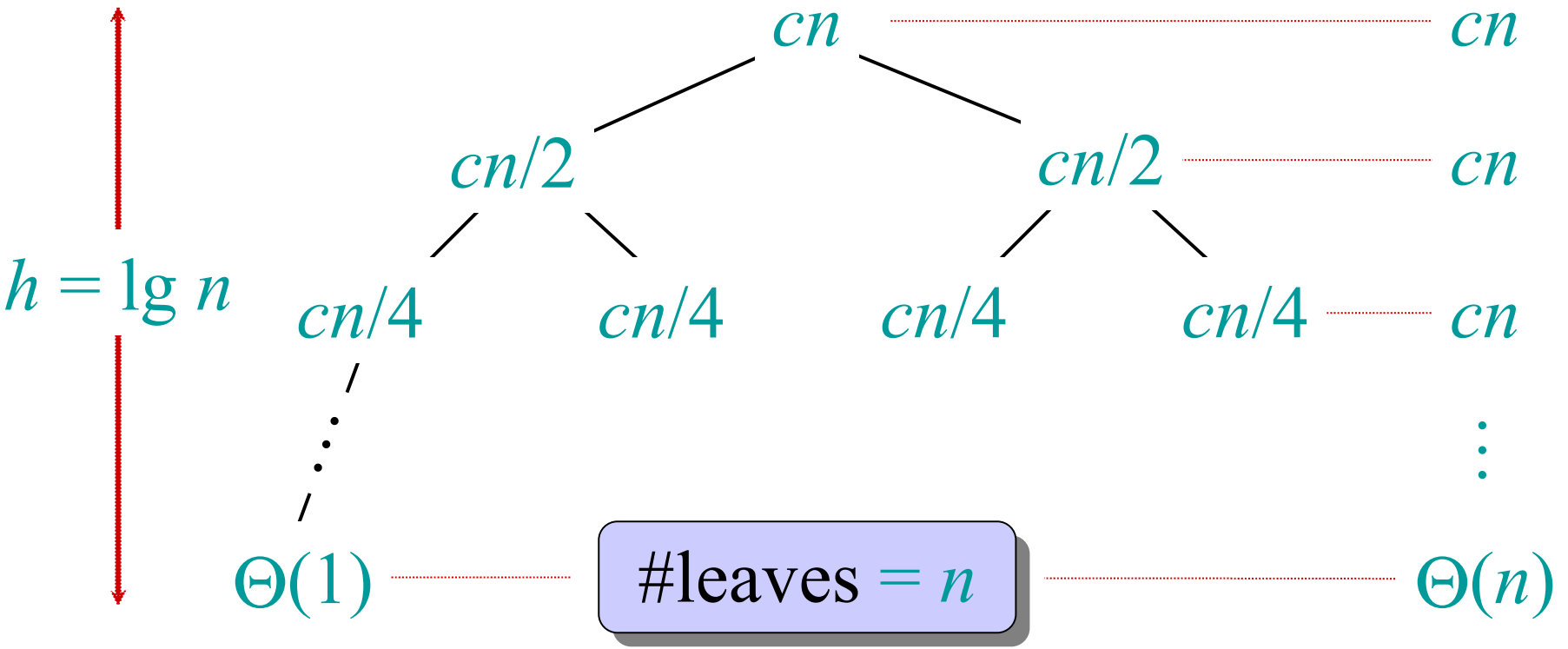

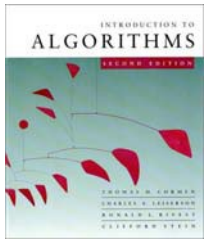

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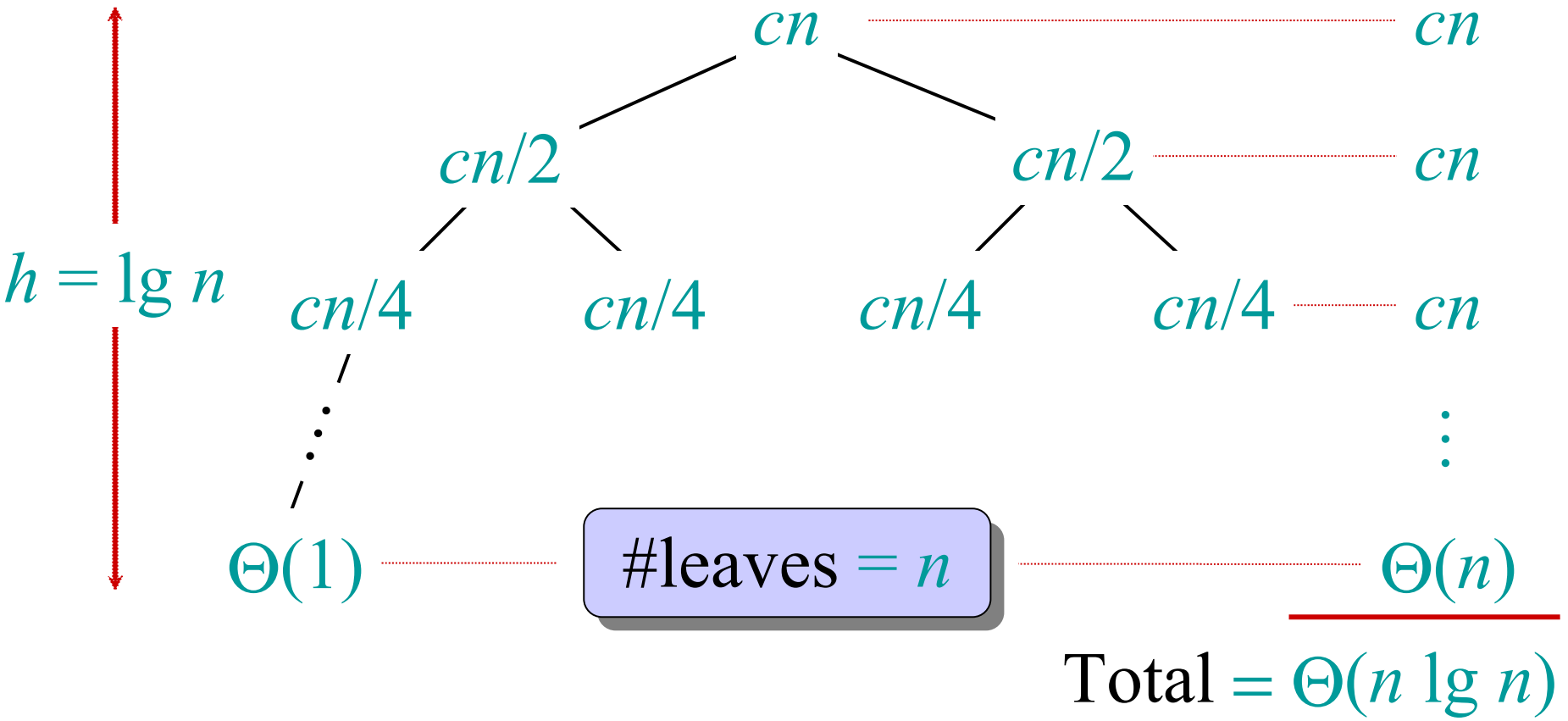
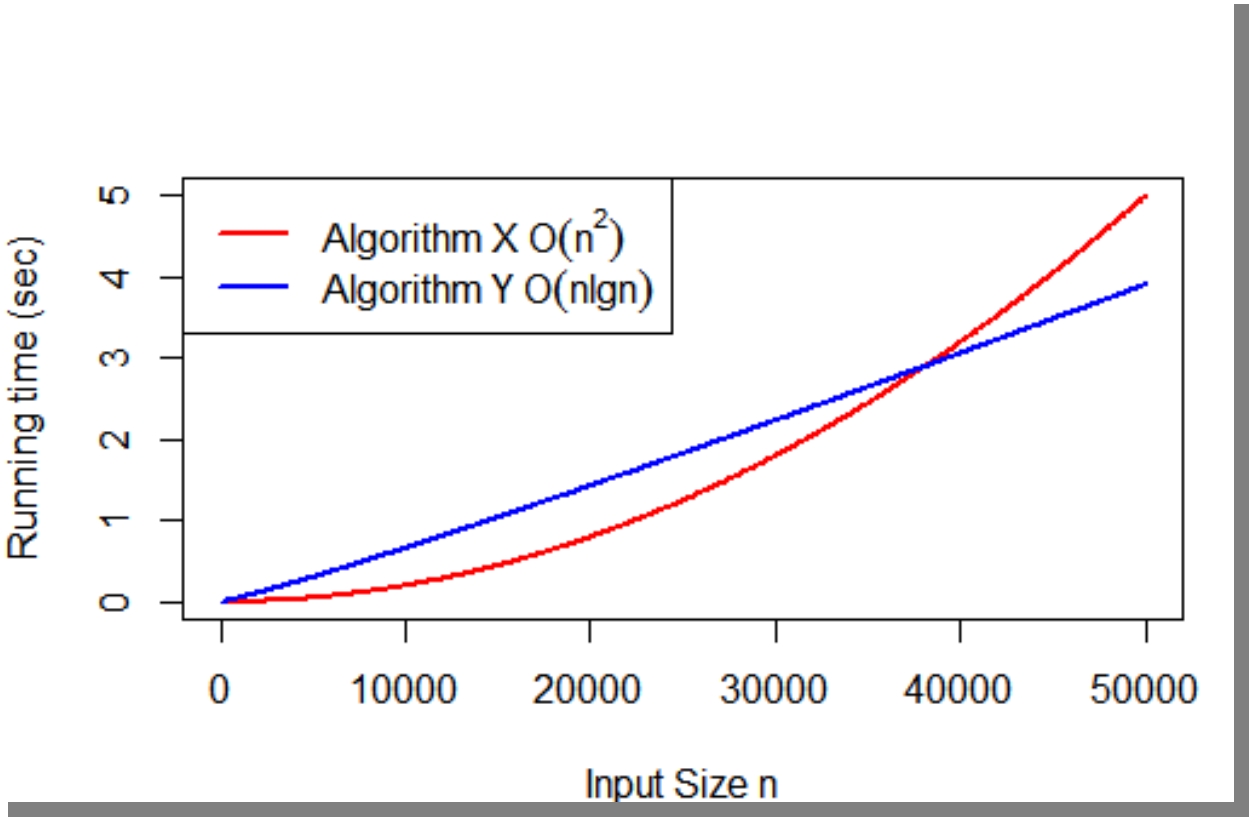


Merge Sort Analysis

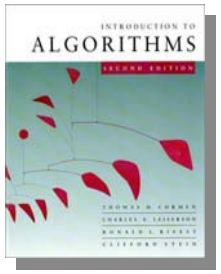
- What about space?
- Merge Sort does not sort "in place" as it needs temporary space for the "Merge" subroutine
- It needs space: $S(n) = \Theta(n)$

Conclusion

- Insertion Sort:
  - $T(n) = \Theta(n^2)$
  - $S(n) = \Theta(1)$
- Merge Sort:
  - $T(n) = \Theta(n \lg n)$
  - $S(n) = \Theta(n)$



- Insertion sort better for smaller n, merge sort for larger n

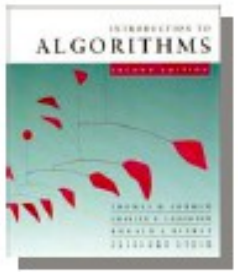


Asymptotic notation

O-notation (upper bounds):

We write $f(n) = O(g(n))$ if there exist constants $c > 0$, $n_0 > 0$ such that $0 \le f(n) \le cg(n)$ for all $n \ge n_0$.

EXAMPLE: $2n^2 = O(n^3)$ $\quad (c = 1, n_0 = 2)$



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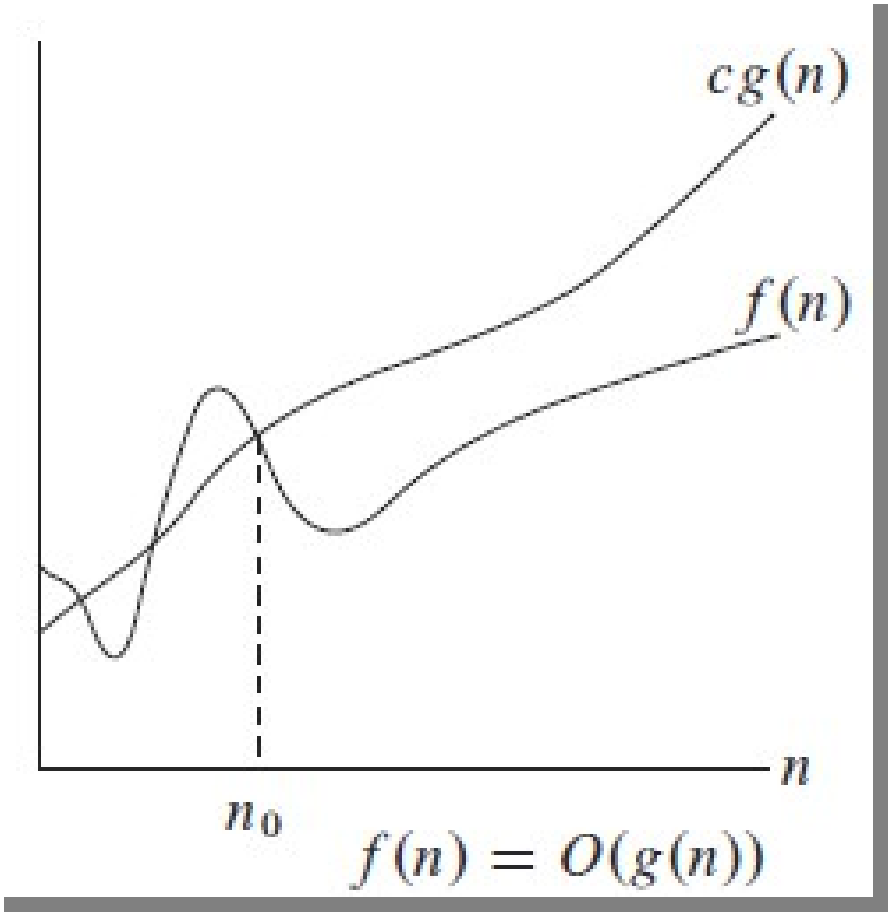

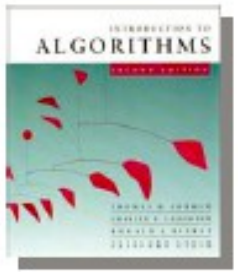

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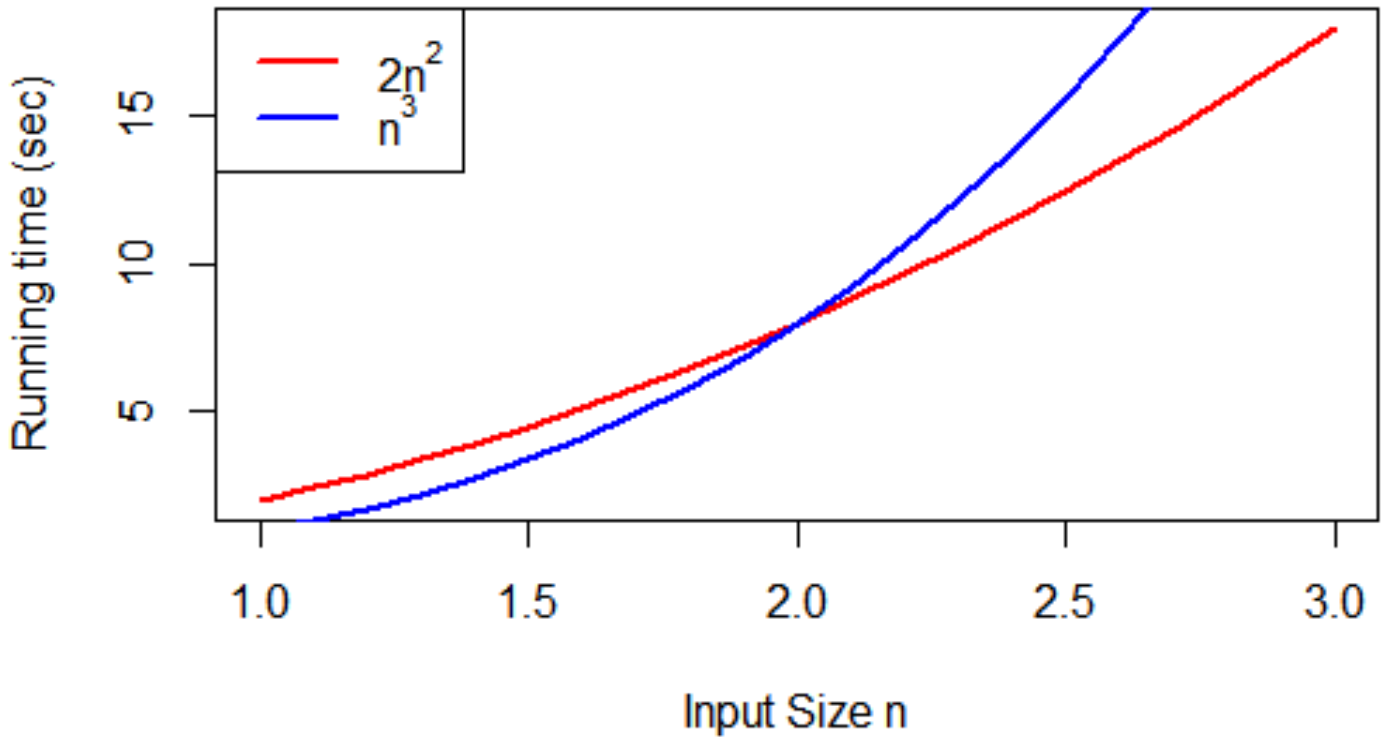

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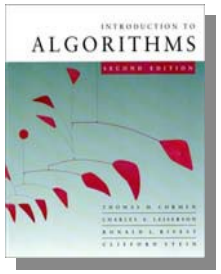

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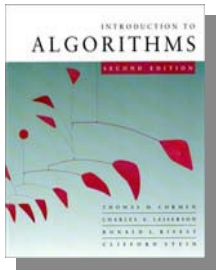
functions, not values



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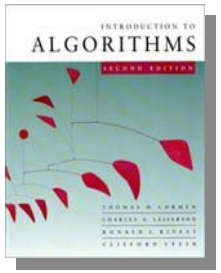
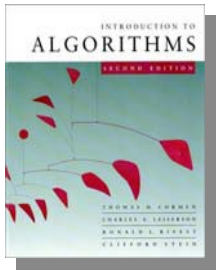
functions, not values

funny, "one-way" equality



Set definition of O-notation

$O(g(n)) = \{ f(n)$: there exist constants $c > 0$, $n_0 > 0$ such that $0 \le f(n) \le cg(n)$ for all $n \ge n_0 \}$



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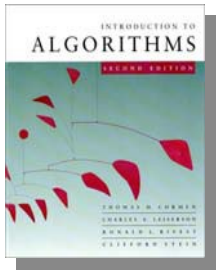
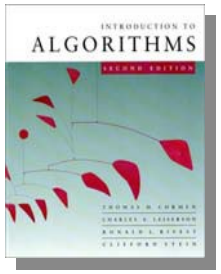


Macro substitution

Convention: A set in a formula represents an anonymous function in the set.



Macro substitution

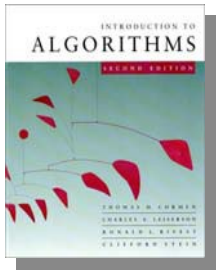
Convention: A set in a formula represents an anonymous function in the set.

EXAMPLE: $f(n) = n^3 + O(n^2)$

means

$f(n) = n^3 + h(n)$

for some $h(n) \in O(n^2)$.



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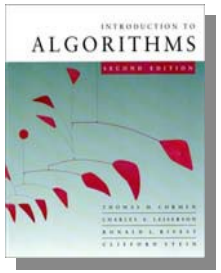
EXAMPLE: $n^2 + O(n) = O(n^2)$

means

for any $f(n) \in O(n)$:

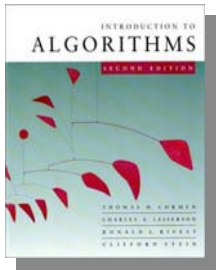
$$n^2 + f(n) = h(n)$$

for some $h(n) \in O(n^2)$.



Ω-notation (lower bounds)

O-notation is an *upper-bound* notation. It makes no sense to say $f(n)$ is at least $O(n^2)$.



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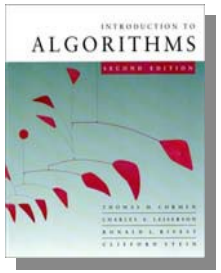

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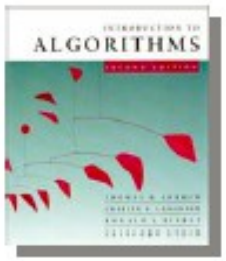
EXAMPLE: $\sqrt{n} = \Omega(\lg n)$ $(c = 1, n_0 = 16)$



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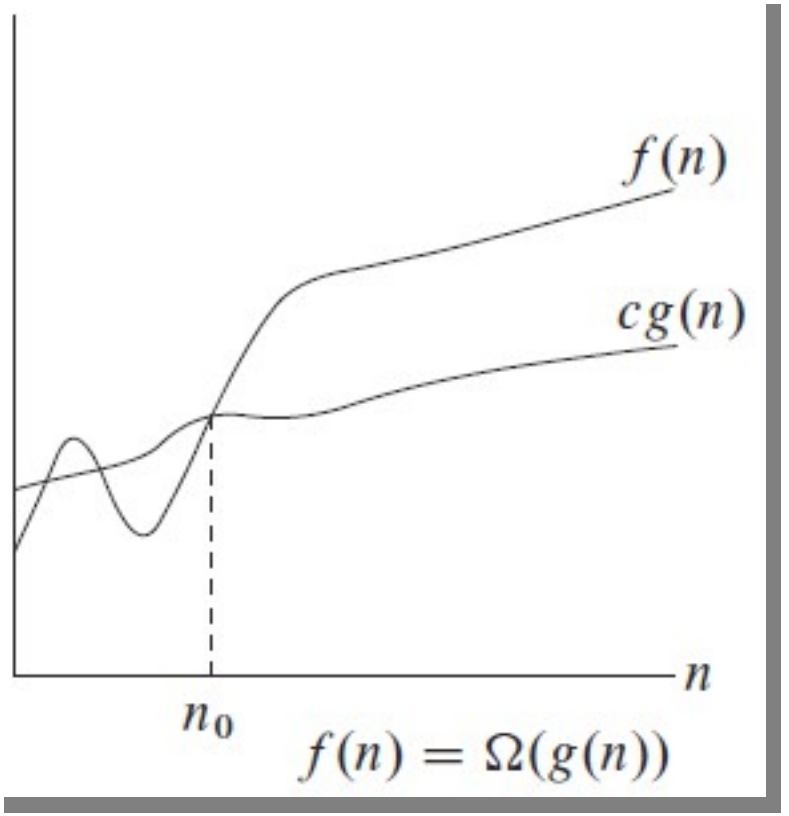

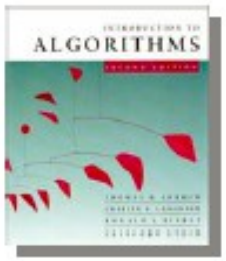

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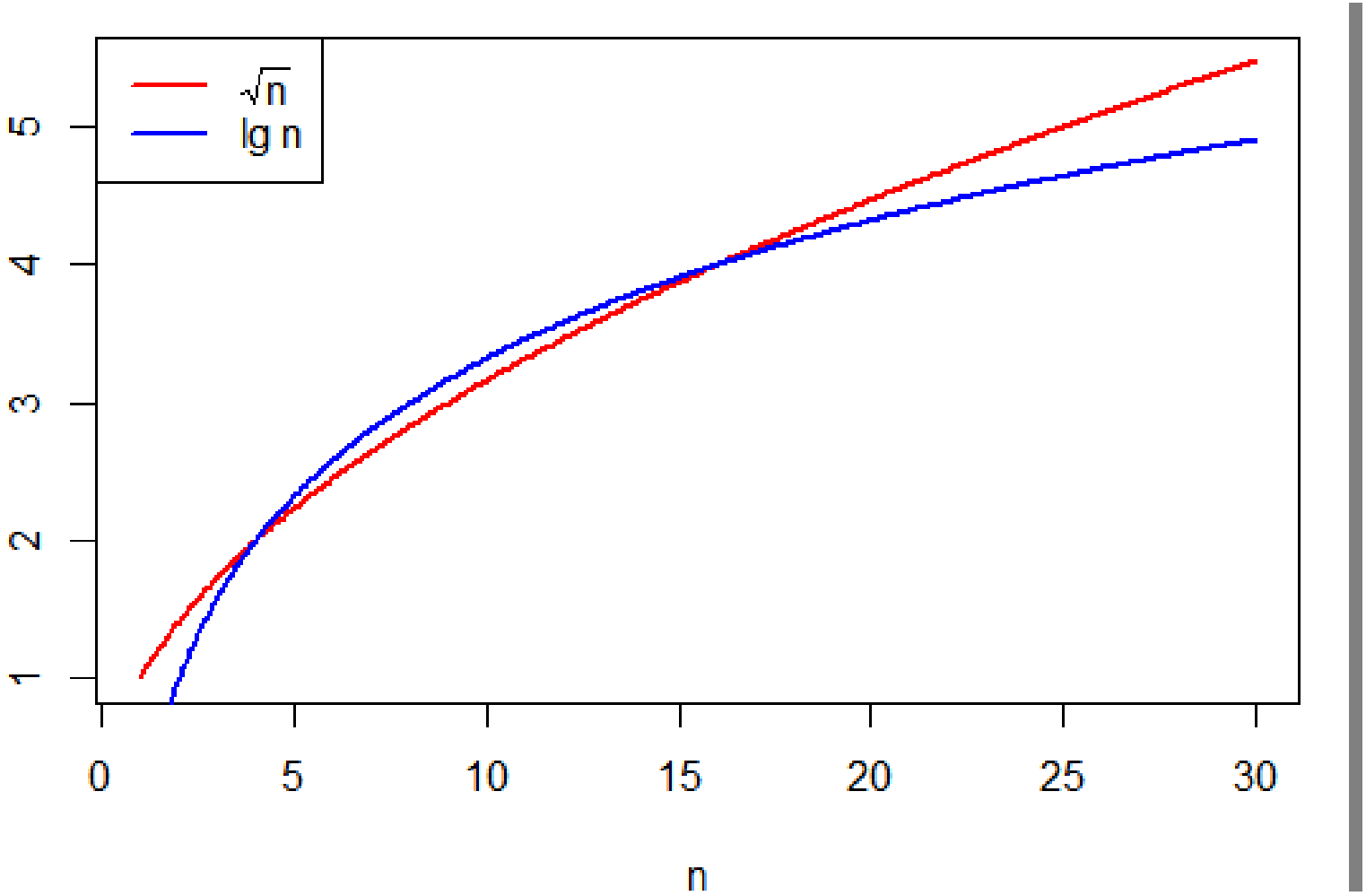

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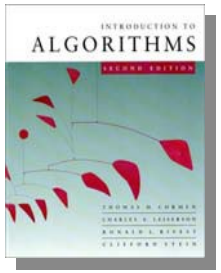
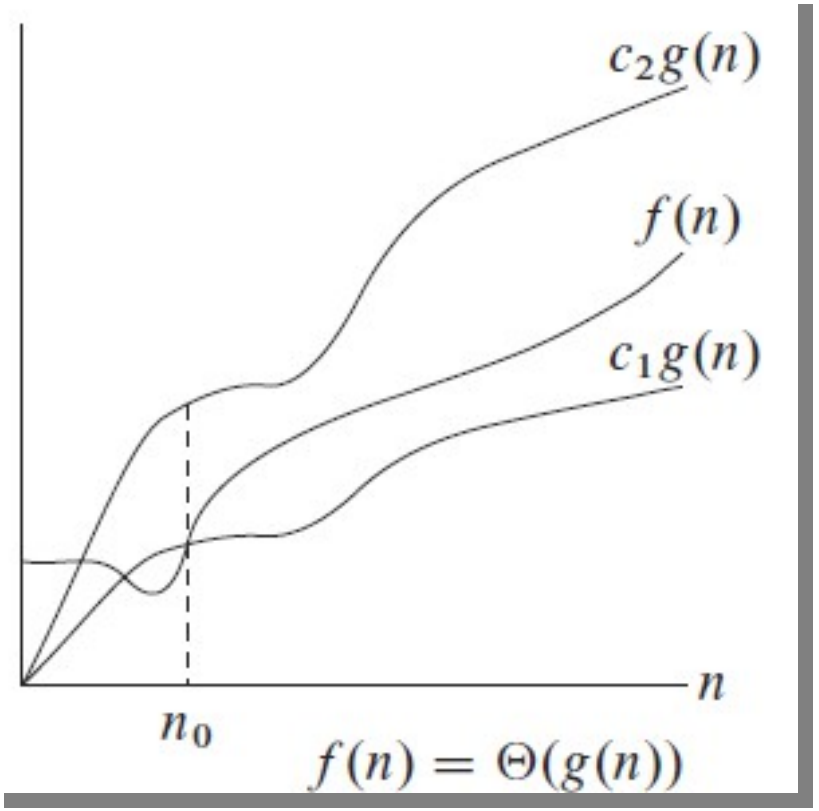
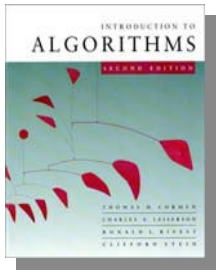


Θ-notation (tight bounds)

$$\Theta(g(n)) = O(g(n)) \cap \Omega(g(n))$$





Θ-notation (tight bounds)

$$\Theta(g(n)) = O(g(n)) \cap \Omega(g(n))$$

EXAMPLE: $\frac{1}{2}n^2 - 2n = \Theta(n^2)$

Recap

- Merge Sort
- Recurrences
- Asymptotic Notation
- Next: More on recurrences and Divide and Conquer