

CMP302: Algorithms



Lecture 10: Dynamic Programming

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Fall 2013

Agenda

- Fibonacci Numbers
- Longest Common Subsequence
- Properties of Dynamic Programming
 - Optimal Substructure
 - Overlapping Subproblems

Acknowledgment

A lot of slides adapted from the slides of Erik Demaine and Charles Leiserson

Fibonacci Numbers

$$F_n = F_{n-1} + F_{n-2}$$

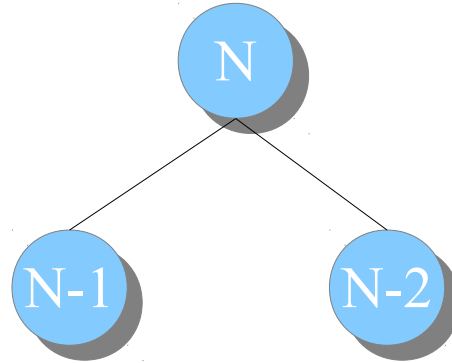
Sequence: 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

Recursive Solution

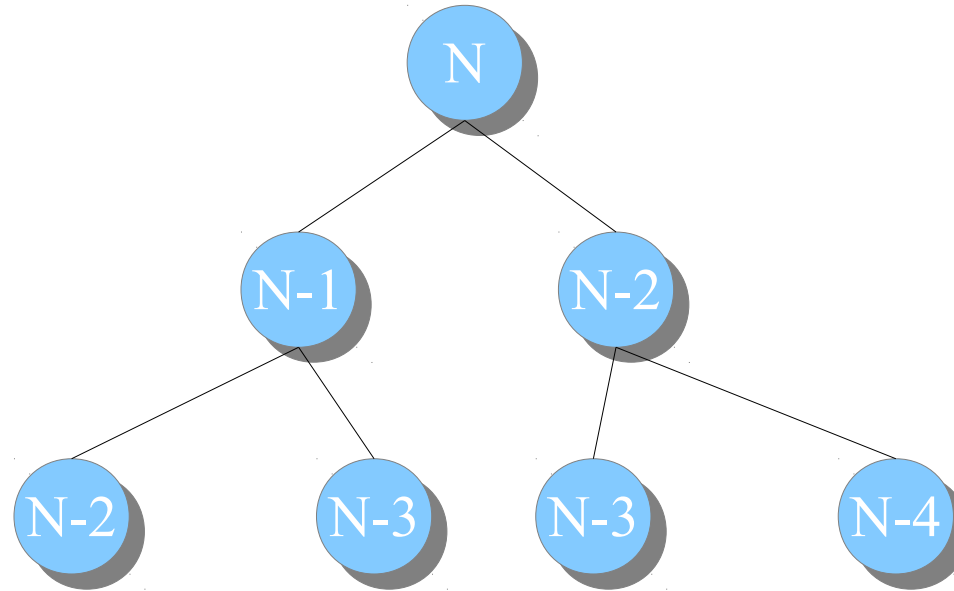
```
int fibonacci(int n) {  
    if (n == 0) return 0;  
    if (n == 1) return 1;  
  
    return fibonacci(n-1) + fibonacci(n-2);  
}
```

Recurrence: $T(n) = T(n-1) + T(n-2)$

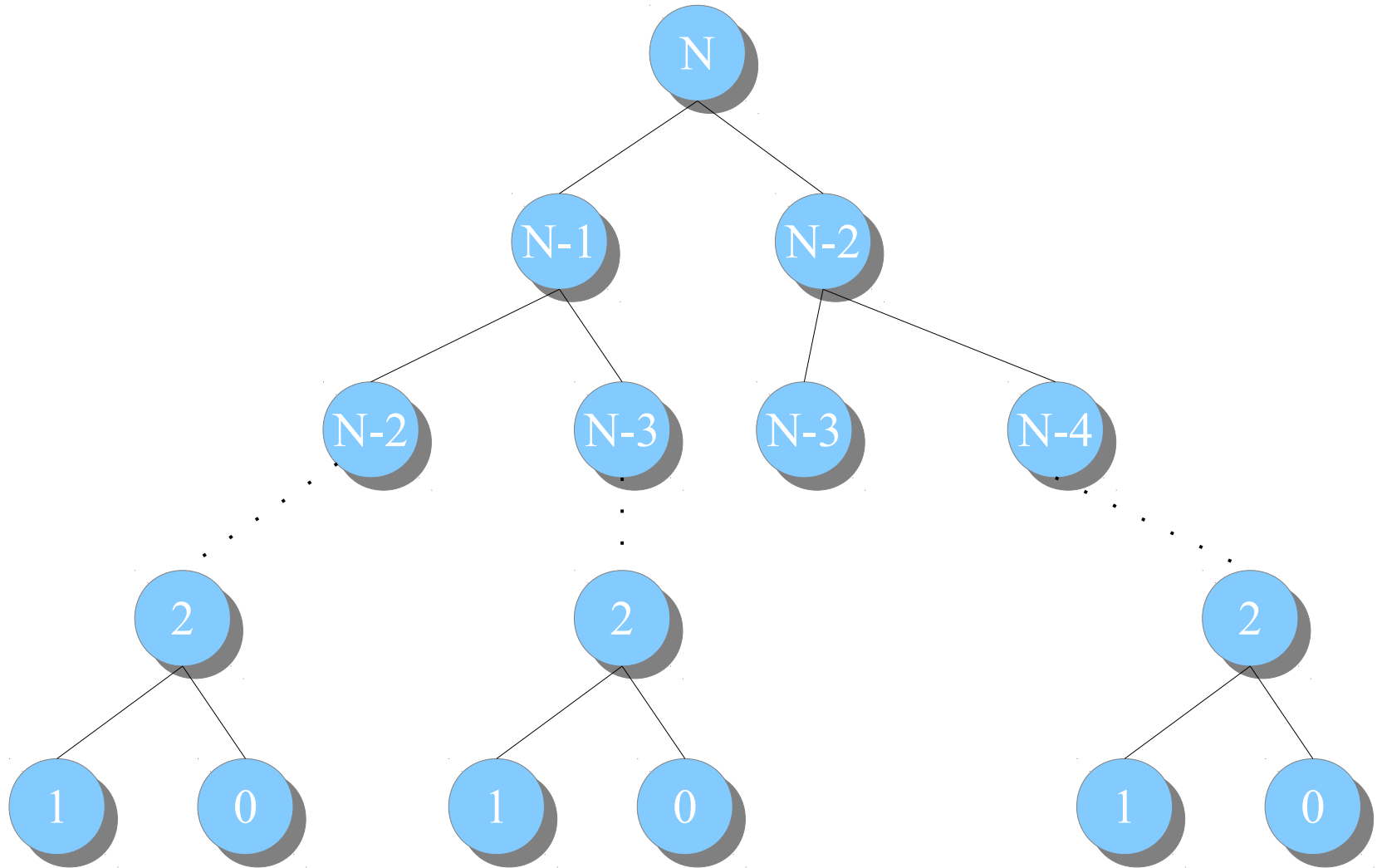
Fibonacci Numbers



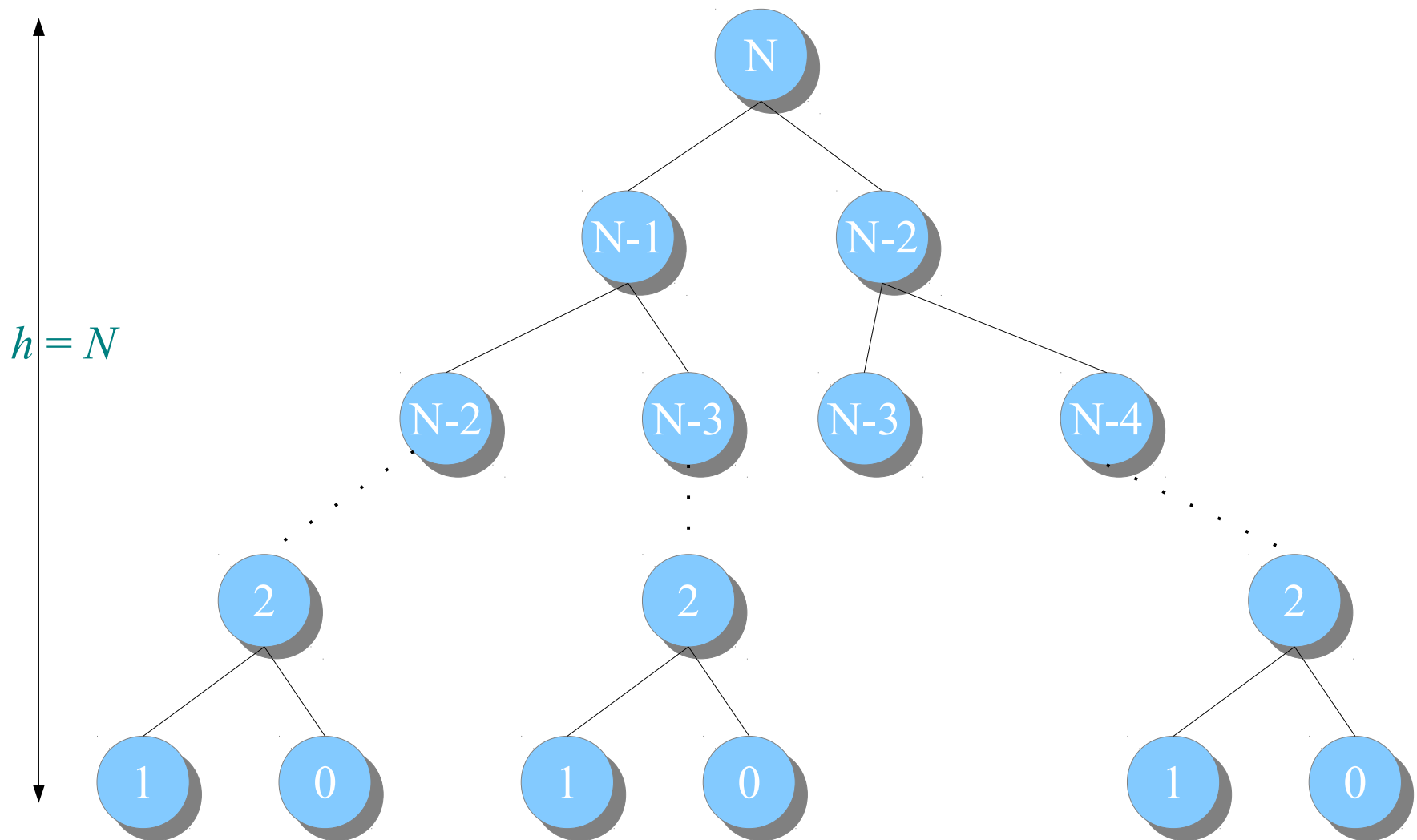
Fibonacci Numbers



Fibonacci Numbers



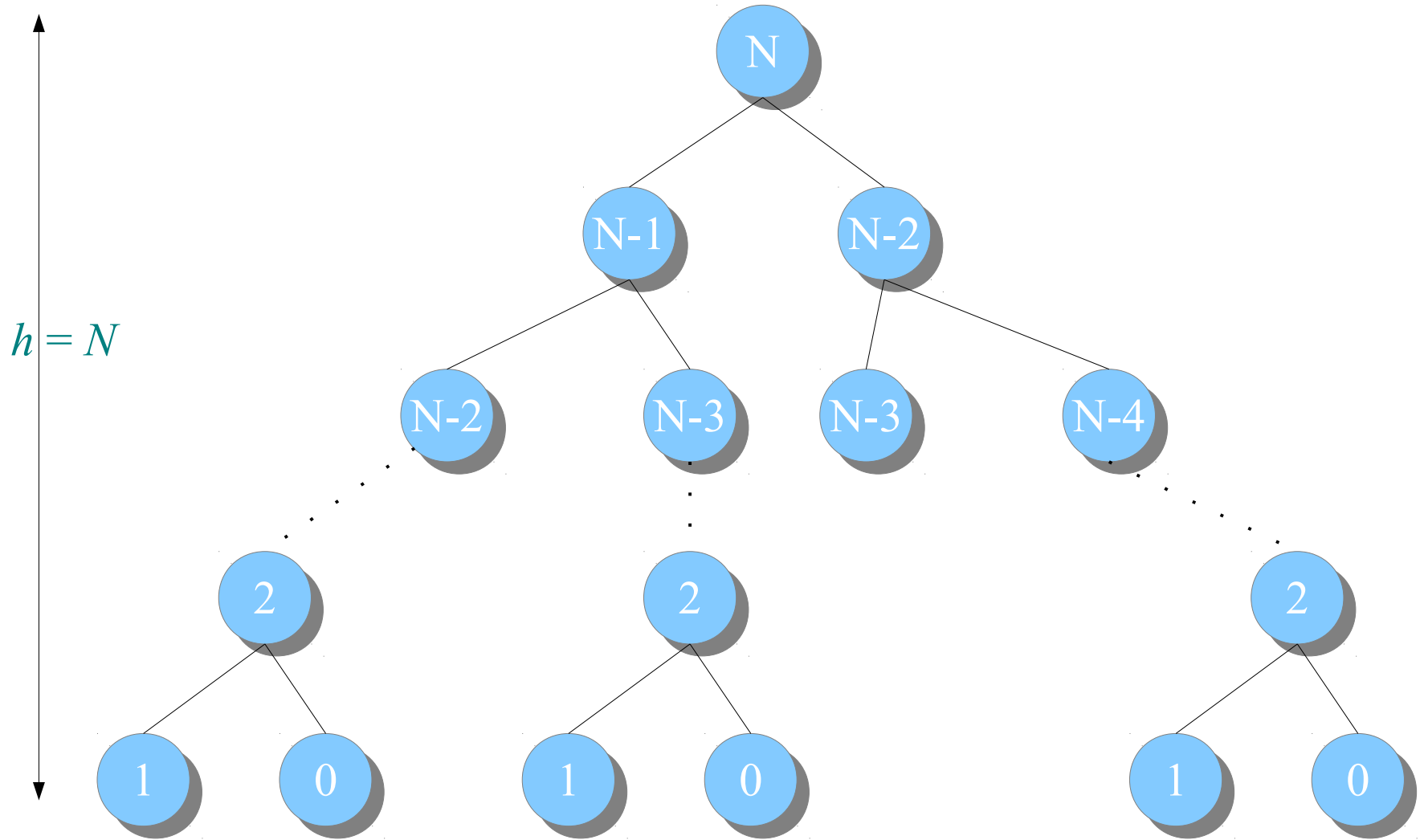
Fibonacci Numbers



Number of leaves $\leq 2^h$

Number of nodes $\leq 2^{h+1} - 1$

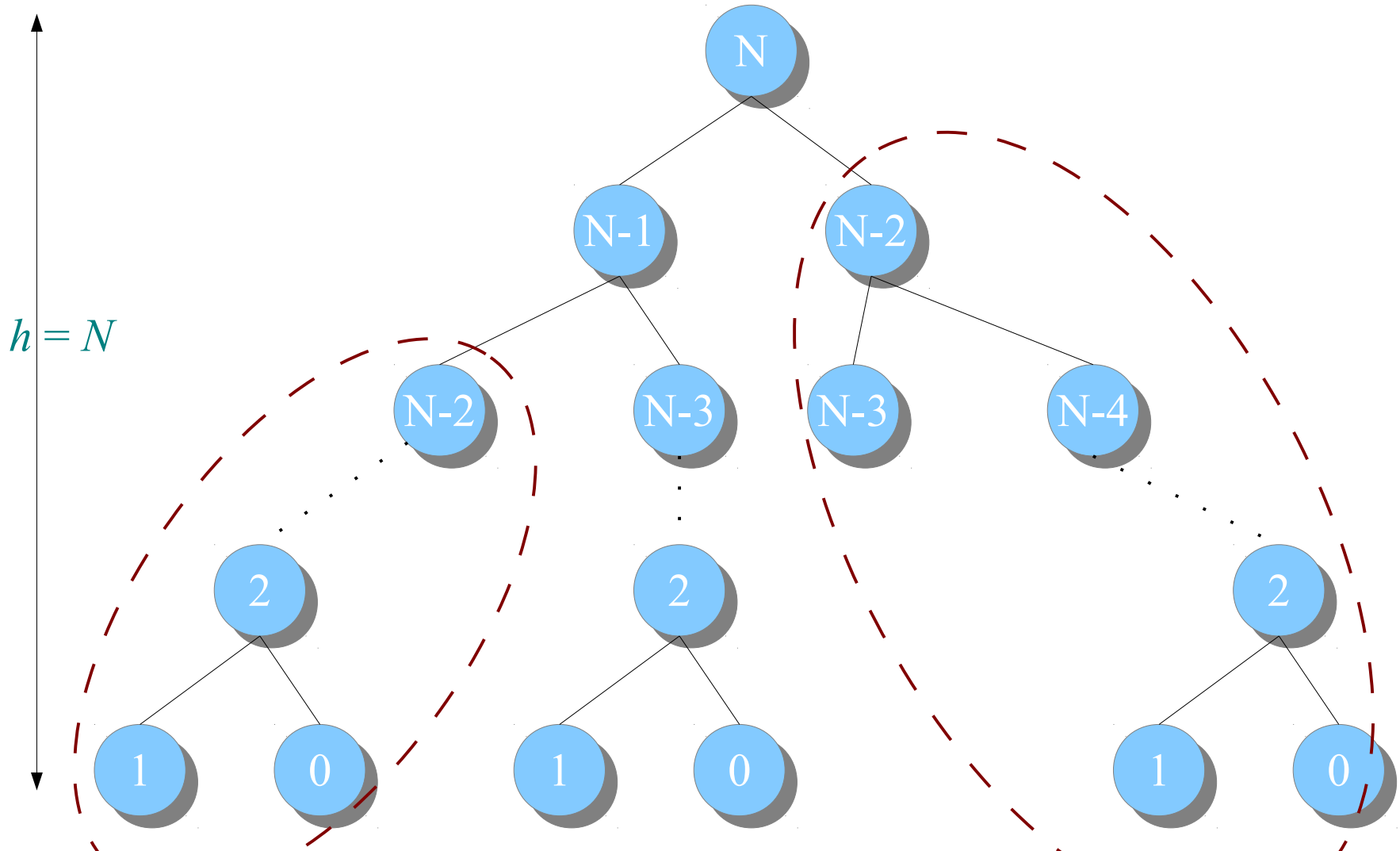
Fibonacci Numbers



$$T(n) = O(2^n)$$

Exponential time !!

Fibonacci Numbers



What's the **problem**? Too many **repeated** subproblems!

Idea: solve subproblems once and store results.

Fibonacci Numbers

$$F_n = F_{n-1} + F_{n-2}$$

Iterative Solution

```
int fibonacci(int n) {  
  
    vector<int> fs(n+1);  
    fs[0] = 0;  
    fs[1] = 1;  
  
    for (int i = 2; i <= n; ++i)  
        fs[i] = fs[i-1] + fs[i-2];  
  
    return fs[n];  
}
```

Linear time !! $T(n) = \Theta(n)$ Trade off space for time.

Fibonacci Numbers

$$F_n = F_{n-1} + F_{n-2}$$

Memoization

Compute values **once**
and use many times

```
vector<int> fibs(MAX_N);  
fill(fibs, -1);  
  
int fibonacci(int n) {  
    if (fibs[n] == -1) {  
        if (n == 0) {  
            fibs[0] = 0;  
        } else if (n == 1) {  
            fibs[1] = 1;  
        } else {  
            fibs[n] = fibonacci(n-2) + fibonacci(n-1);  
        }  
    }  
  
    return fibs[n];  
}
```

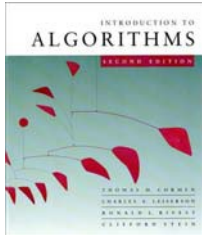
Recursive with
time $\Theta(n)$

Fibonacci Numbers

- Key Observation:
 - Lots of small subproblems
 - Subproblems are repeated over and over
- Key Idea:
 - Solve each subproblem only once
 - Store solution and reuse many times

Dynamic Programming

- Design technique
- Can be used to:
 - Make recursive solutions more *efficient* by *memoization* or conversion to *iterative* solutions
 - Solve *optimization* problems *efficiently* e.g. find the shortest path, the best matrix parenthesization, the longest common subsequence, ... etc.

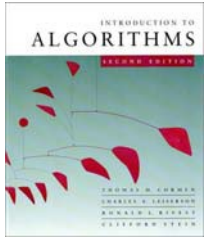


Dynamic programming

Design technique, like divide-and-conquer.

Example: Longest Common Subsequence (LCS)

- Given two sequences $x[1 \dots m]$ and $y[1 \dots n]$, find a longest subsequence common to them both.



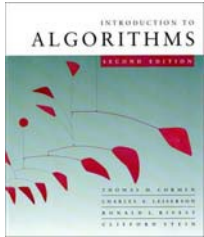
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↖ “a” *not* “the”



Dynamic programming

Design technique, like divide-and-conquer.

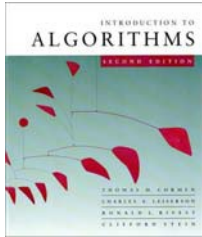
Example: Longest Common Subsequence (LCS)

- Given two sequences $x[1 \dots m]$ and $y[1 \dots n]$, find a longest subsequence common to them both.

“a” *not* “the”

x : A B C B D A B

y : B D C A B A



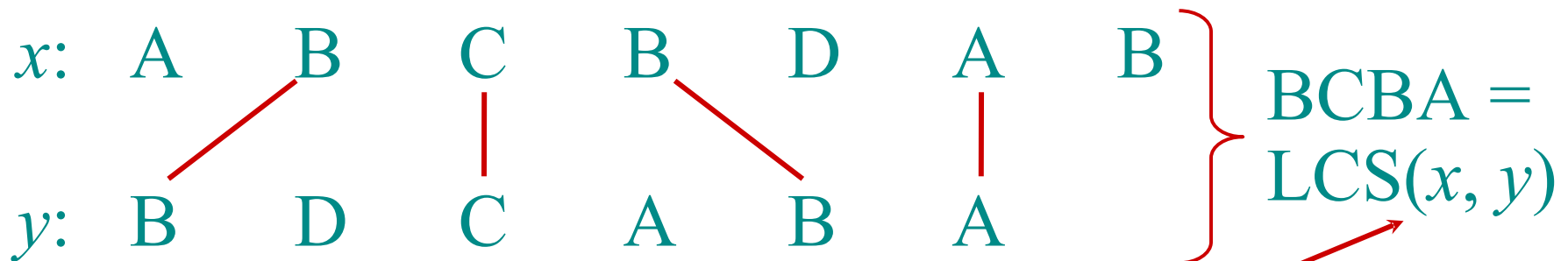
Dynamic programming

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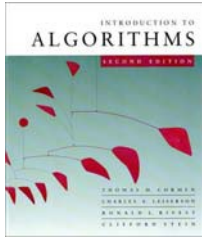
Example: *Longest Common Subsequence (LCS)*

- Given two sequences $x[1..m]$ and $y[1..n]$, find a longest subsequence common to them both.

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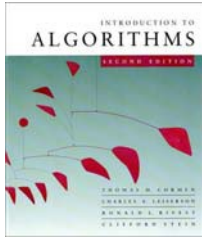


functional notation,
but not a function



Brute-force LCS algorithm

Check every subsequence of $x[1 \dots m]$ to see if it is also a subsequence of $y[1 \dots n]$.



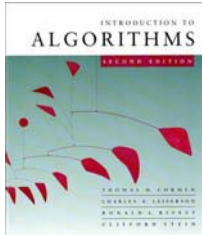
Brute-force LCS algorithm

Check every subsequence of $x[1..m]$ to see if it is also a subsequence of $y[1..n]$.

Analysis

- Checking = $O(n)$ time per subsequence.
- 2^m subsequences of x (each bit-vector of length m determines a distinct subsequence of x).

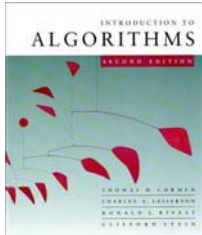
Worst-case running time = $O(n2^m)$
= exponential time.



Towards a better algorithm

Simplification:

1. Look at the *length* of a longest-common subsequence.
2. Extend the algorithm to find the LCS itself.

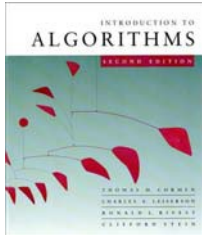


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Notation: Denote the length of a sequence s by $|s|$.



Towards a better algorithm

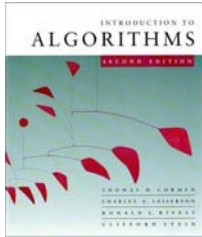
Simplification:

1. Look at the *length* of a longest-common subsequence.
2. Extend the algorithm to find the LCS itself.

Notation: Denote the length of a sequence s by $|s|$.

Strategy: Consider *prefixes* of x and y .

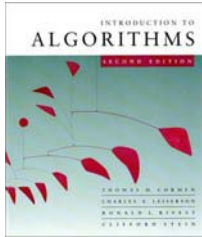
- Define $c[i, j] = |\text{LCS}(x[1 \dots i], y[1 \dots j])|$.
- Then, $c[m, n] = |\text{LCS}(x, y)|$.



Recursive formulation

Theorem.

$$c[i, j] = \begin{cases} c[i-1, j-1] + 1 & \text{if } x[i] = y[j], \\ \max \{c[i-1, j], c[i, j-1]\} & \text{otherwise.} \end{cases}$$

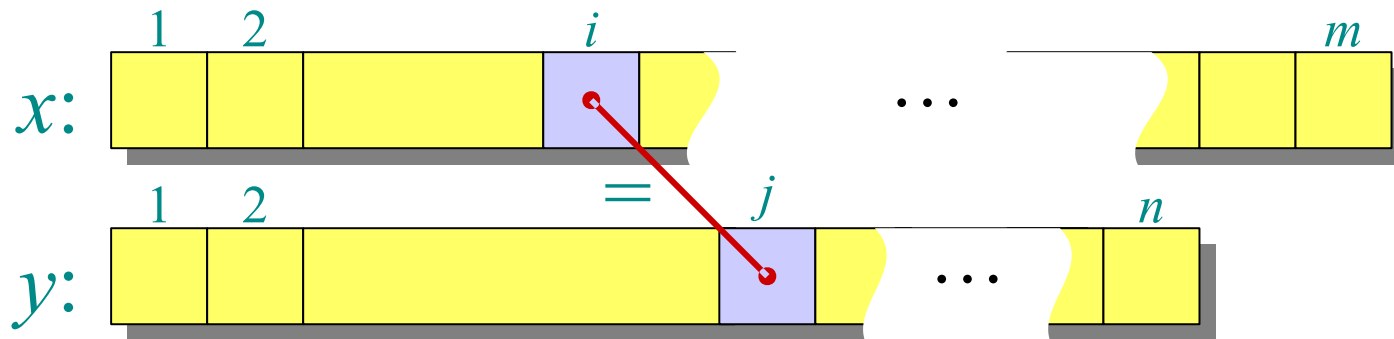


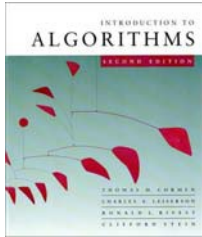
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Proof. Case $x[i] = y[j]$:



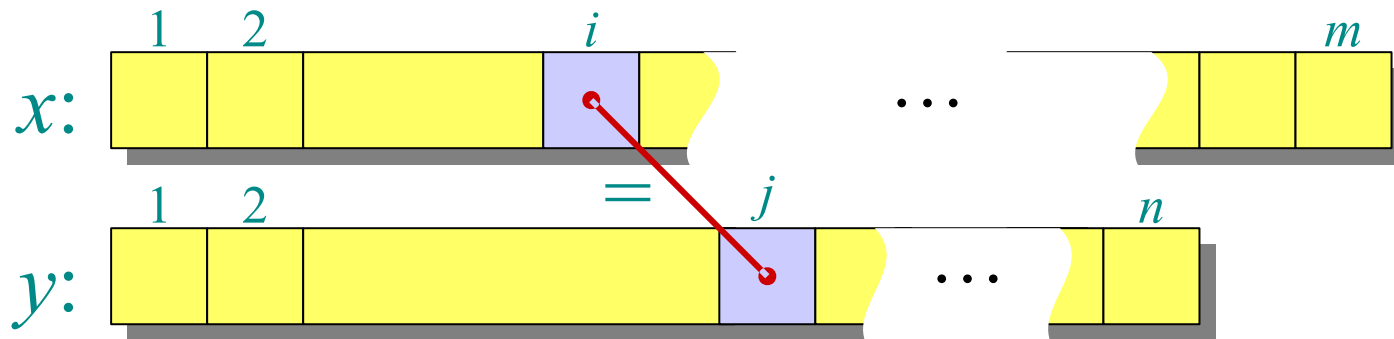


Recursive formulation

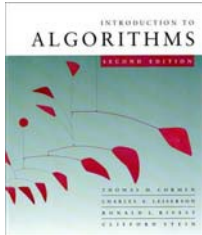
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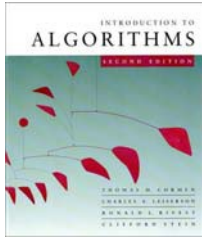
Let $z[1 \dots k] = \text{LCS}(x[1 \dots i], y[1 \dots j])$, where $c[i, j] = k$. Then, $z[k] = x[i]$, or else z could be extended. Thus, $z[1 \dots k-1]$ is CS of $x[1 \dots i-1]$ and $y[1 \dots j-1]$.



Proof (continued)

Claim: $z[1 \dots k-1] = \text{LCS}(x[1 \dots i-1], y[1 \dots j-1])$.

Suppose w is a longer CS of $x[1 \dots i-1]$ and $y[1 \dots j-1]$, that is, $|w| > k-1$. Then, **cut and paste**: $w || z[k]$ (w concatenated with $z[k]$) is a common subsequence of $x[1 \dots i]$ and $y[1 \dots j]$ with $|w || z[k]| > k$. Contradiction, proving the claim.



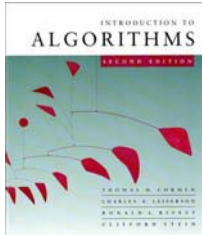
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Thus, $c[i-1, j-1] = k-1$, which implies that $c[i, j] = c[i-1, j-1] + 1$.

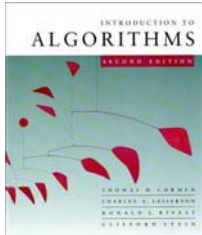
Other cases are similar. □



Dynamic-programming hallmark #1

Optimal substructure

*An optimal solution to a problem
(instance) contains optimal
solutions to subproblems.*

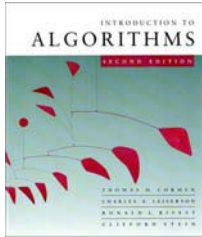


Dynamic-programming hallmark #1

Optimal substructure

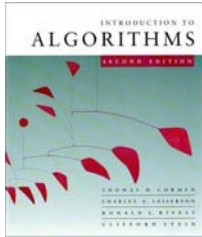
An optimal solution to a problem (instance) contains optimal solutions to subproblems.

If $z = \text{LCS}(x, y)$, then any prefix of z is an LCS of a prefix of x and a prefix of y .



Recursive algorithm for LCS

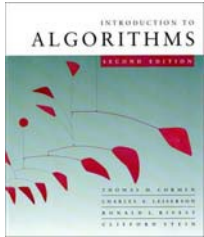
```
LCS( $x, y, i, j$ )  
  if  $x[i] = y[j]$   
    then  $c[i, j] \leftarrow \text{LCS}(x, y, i-1, j-1) + 1$   
    else  $c[i, j] \leftarrow \max \{ \text{LCS}(x, y, i-1, j),$   
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```



Recursive algorithm for LCS

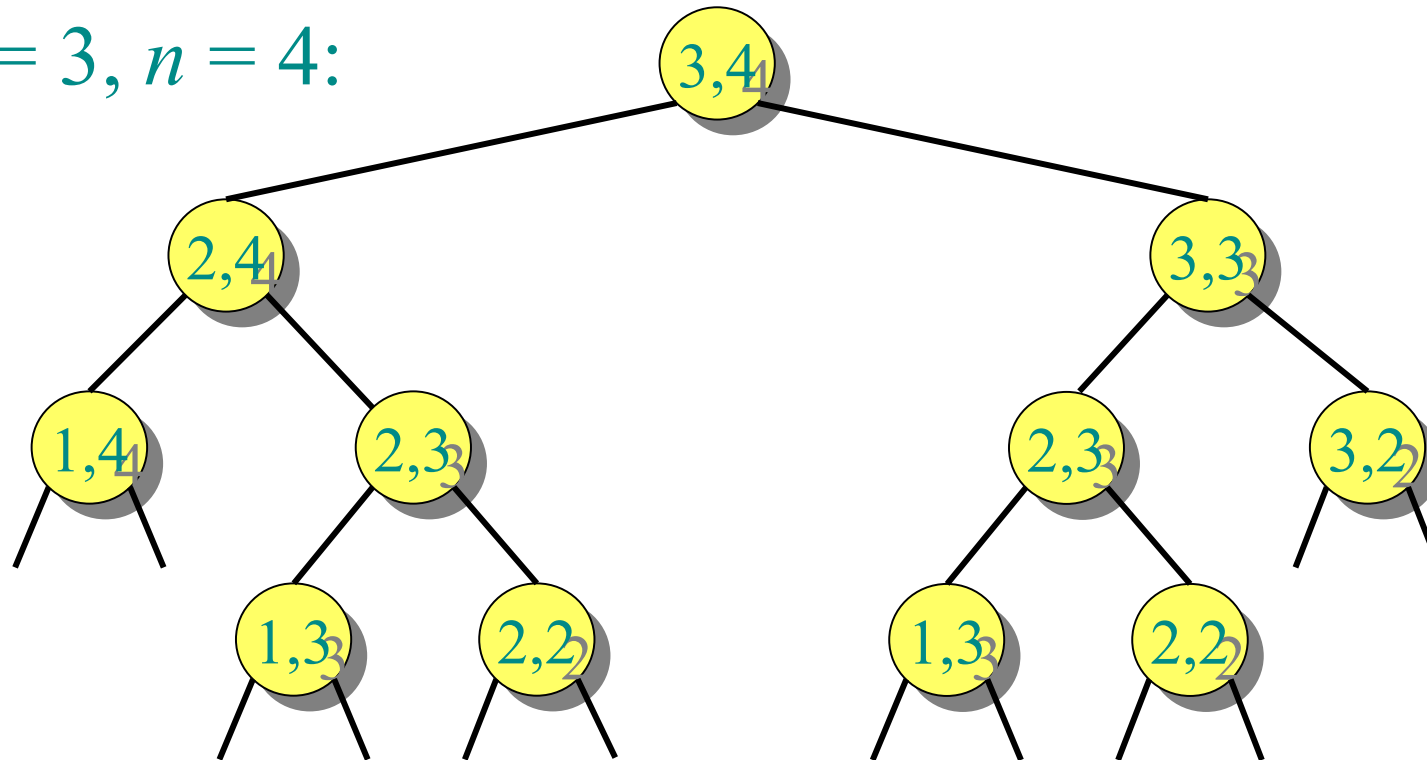
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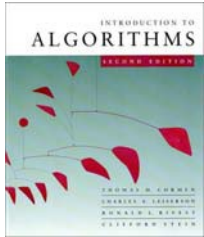
Worst-case: $x[i] \neq y[j]$, in which case the algorithm evaluates two subproblems, each with only one parameter decremented.



Recursion tree

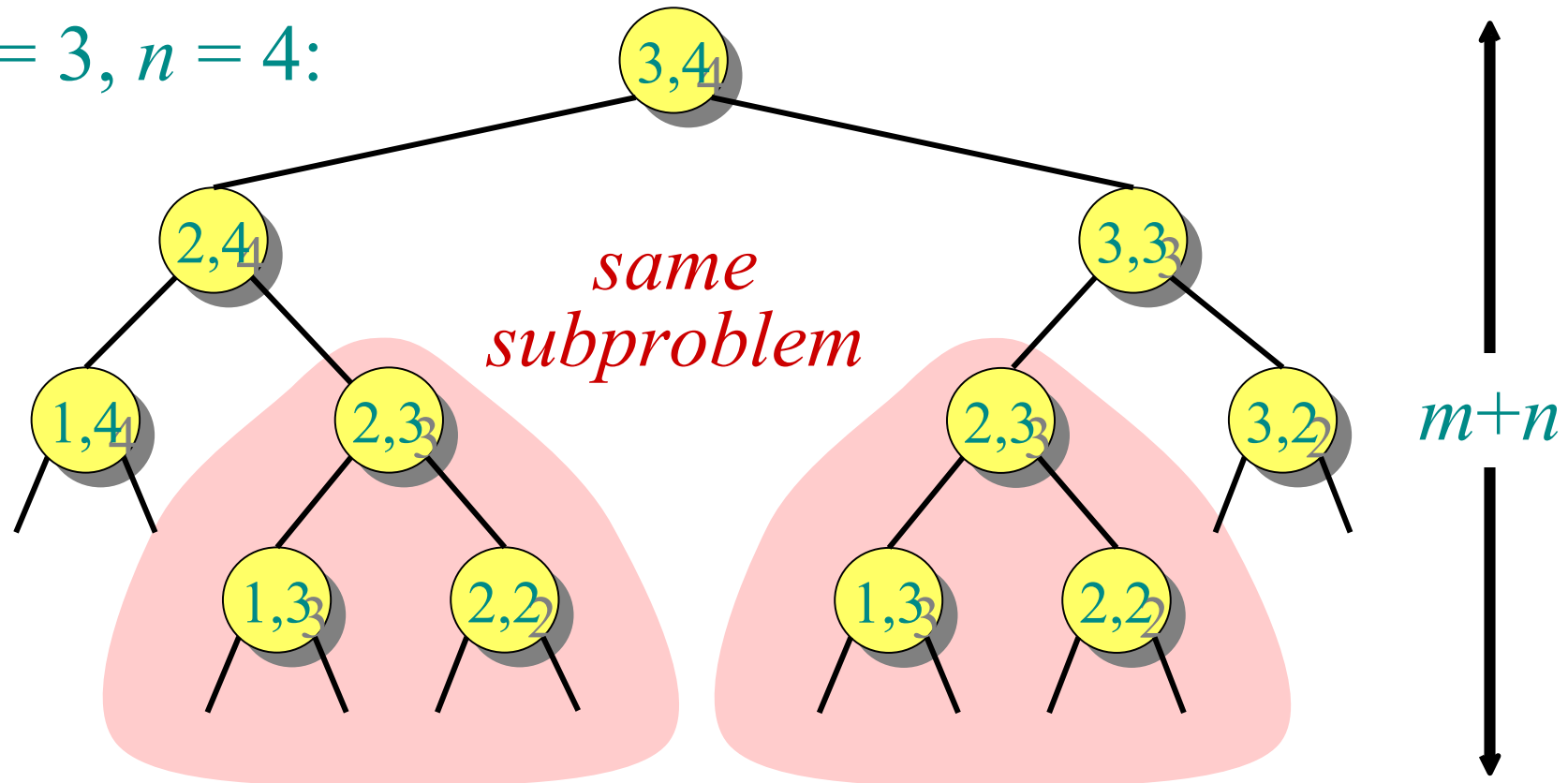
$m = 3, n = 4$:



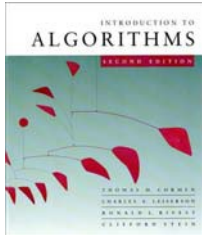


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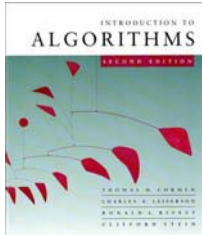
Height = $m + n \Rightarrow$ work potentially exponential, but we're solving subproblems already solved!



Dynamic-programming hallmark #2

Overlapping subproblems

A recursive solution contains a “small” number of distinct subproblems repeated many times.

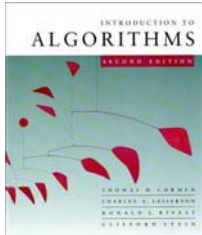


Dynamic-programming hallmark #2

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The number of distinct LCS subproblems for two strings of lengths m and n is only mn .



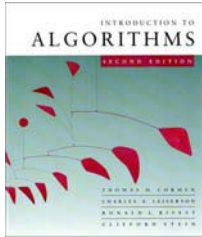
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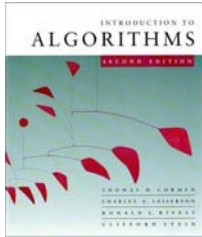
The number of distinct LCS subproblems for two strings of lengths m and n is only mn .

All the possible combination of prefixes of x and y



Memoization algorithm

Memoization: After computing a solution to a subproblem, store it in a table. Subsequent calls check the table to avoid redoing work.



Memoization algorithm

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$LCS(x, y, i, j)$

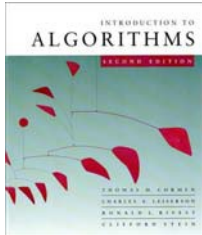
if $c[i, j] = \text{NIL}$

then if $x[i] = y[j]$

then $c[i, j] \leftarrow LCS(x, y, i-1, j-1) + 1$

else $c[i, j] \leftarrow \max \{ LCS(x, y, i-1, j),$
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} *same as before*



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} *same
as
before*

Time = $\Theta(mn)$ = constant work per table entry.

Space = $\Theta(mn)$.

Dynamic Programming Algorithm

Bottom up algorithm

```
LCS-Length(X, Y, m, n)
  for i = 1 to m
    c[i, 0] = 0
  for j = 1 to n
    c[0, j] = 0

  for i = 1 to m
    for j = 1 to n
      if X[i] == Y[j]
        c[i, j] = c[i-1, j-1] + 1
      elseif c[i-1, j] > c[i, j-1]
        c[i, j] = c[i-1, j]
      else
        c[i, j] = c[i, j-1]

  return c[m, n]
```

Initialize base cases

Loop in row-major order

Case 1

Case 2 & 3

Dynamic Programming Algorithm

```
for i = 1 to m  
  c[i, 0] = 0  
  
for j = 1 to n  
  c[0, j] = 0
```

	A	B	C	B	D	A	B
	0	0	0	0	0	0	0
B	0						
D	0						
C	0						
A	0						
B	0						
A	0						

Dynamic Programming Algorithm

```
for i = 1 to m
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    else
      c[i, j] = c[i, j-1]
```

	A	B	C	B	D	A	B
B	0	0	0	0	0	0	0
D	0						
C	0						
A	0						
B	0						
A	0						

Dynamic Programming Algorithm

```
for i = 1 to m
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    else
      c[i, j] = c[i, j-1]
```

	A	B	C	B	D	A	B
	0	0	0	0	0	0	0
B	0	0	1	1	1	1	1
D	0	0	1	1	1	2	2
C	0						
A	0						
B	0						
A	0						

Dynamic Programming Algorithm

```
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	A	B	C	B	D	A	B	
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B	0	0	1	1	1	1	1	
D	0	0	1	1	1	2	2	
C	0	0	1	2	2	2	2	
A	0	1	1	2	2	2	3	3
B	0	1	2	2	3	3	3	4
A	0	1	2	2	3	3	4	4

Dynamic Programming Algorithm

```

for i = 1 to m
  for j = 1 to n
    if X[i] == Y[j]
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      c[i, j] = c[i-1, j]
    else
      c[i, j] = c[i, j-1]
  
```

Time = $\Theta(mn)$

Space = $\Theta(mn)$

		A	B	C	B	D	A	B
B	0	0	0	0	0	0	0	0
D	0	0	1	1	1	2	2	2
C	0	0	1	2	2	2	2	2
A	0	1	1	2	2	2	3	3
B	0	1	2	2	3	3	3	4
A	0	1	2	2	3	3	4	4

Dynamic Programming Algorithm

Reconstruct the LCS by tracing backwards through the table

	A	B	C	B	D	A	B
	0	0	0	0	0	0	0
B	0	0	1	1	1	1	1
D	0	0	1	1	1	2	2
C	0	0	1	2	2	2	2
A	0	1	1	2	2	2	3
B	0	1	2	2	3	3	4
A	0	1	2	2	3	3	4

Dynamic Programming Algorithm

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Dynamic Programming Algorithm

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	A	B	C	B	D	A	B
	0	0	0	0	0	0	0
B	0	0	1	1	1	1	1
D	0	0	1	1	1	2	2
C	0	0	1	2	2	2	2
A	0	1	1	2	2	2	3
B	0	1	2	2	3	3	4
A	0	1	2	2	3	3	4

Dynamic Programming Algorithm

Reconstruct the LCS by tracing backwards through the table

	A	B	C	B	D	A	B
	0	0	0	0	0	0	0
B	0	0	1	1	1	1	1
D	0	0	1	1	1	2	2
C	0	0	1	2	2	2	2
A	0	1	1	2	2	2	3
B	0	1	2	2	3	3	4
A	0	1	2	2	3	3	4

Dynamic Programming Algorithm

Reconstruct the LCS by tracing backwards through the table

LCS = “**BCBA**”

	A	B	C	B	D	A	B
	0	0	0	0	0	0	0
B	0	0	1	1	1	1	1
D	0	0	1	1	1	2	2
C	0	0	1	2	2	2	2
A	0	1	1	2	2	2	3
B	0	1	2	2	3	3	4
A	0	1	2	2	3	3	4

Edit Distance

The algorithm can be used to compute the **Edit Distance** between two strings i.e. number of insertions, deletions, and substitutions.

Alignment

AB*C*BDAB
 *BDCAB*A*

Deletion
Insertion
Match

		A	B	C	B	D	A	B
		0	0	0	0	0	0	0
B		0	0	1	1	1	1	1
D		0	0	1	1	1	2	2
C		0	0	1	2	2	2	2
A		0	1	1	2	2	2	3
B		0	1	2	2	3	3	3
A		0	1	2	2	3	3	4

Properties of Dynamic Programming

- Optimal Substructure
 - A solution to the problem is composed of solutions to smaller subproblems
 - At each step we need to examine more than one smaller subproblem (e.g. F_{n-1} and F_{n-2} for **Fibonacci** numbers or $c[i-1, j-1]$, $c[i-1, j]$ and $c[i, j-1]$ for **LCS**)
 - We need to combine their solutions to arrive at the solution to the current problem (e.g. addition or maximum)

Properties of Dynamic Programming

- Overlapping Subproblems
 - Usually there are exponentially many subproblems to solve in the recursive solution (e.g. Fibonacci or LCS)
 - However, these subproblems are not distinct. They keep repeating
 - Two ways to utilize this:
 - **Memoization**: modify the recursive algorithm by first looking up the value before making a recursive call (i.e. **top-down** strategy)
 - **Dynamic Programming**: by computing solutions to the small subproblems first and then combine them to arrive at a solution to the main problem (i.e. **bottom-up** strategy)

Recap

- Fibonacci Numbers
- Longest Common Subsequence
- Properties of Dynamic Programming
 - Optimal Substructure
 - Overlapping Subproblems
- Read Ch. 15 of **CLRS** for more examples (rod cutting, matrix chain multiplication)
- Next:
 - Greedy Algorithms