CMP461: Algorithms

Lecture 15: Dijkstra's Algorithm

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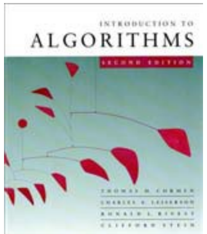
Fall 2013

Agenda

- Properties of shortest paths
- Dijkstra's Algorithm
- Correctness
- Analysis
- Breadth-First Search

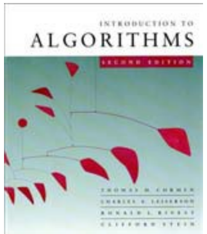
Acknowledgment
A lot of slides adapted from the slides of Erik Demaine and Charles Leiserson.



Paths in graphs

Consider a digraph $G = (V, E)$ with edge-weight function $w : E \rightarrow \mathbb{R}$. The *weight* of path $p = v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_k$ is defined to be

$$w(p) = \sum_{i=1}^{k-1} w(v_i, v_{i+1}) .$$



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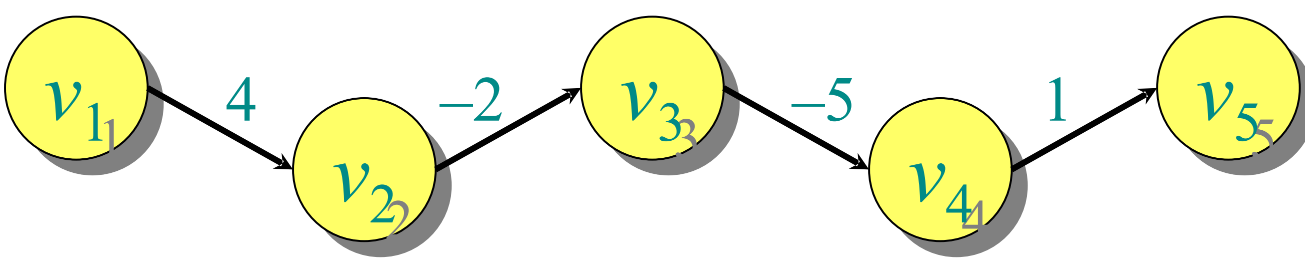
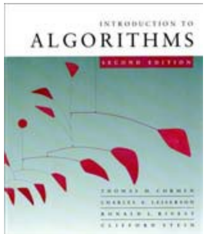
Example:



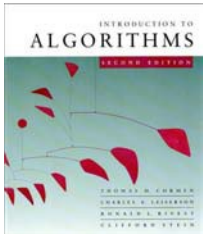
$w(p) = -2$



Shortest paths

A *shortest path* from u to v is a path of minimum weight from u to v. The *shortest-path weight* from u to v is defined as

$$\delta(u, v) = \min\{w(p) : p \text{ is a path from } u \text{ to } v\}.$$



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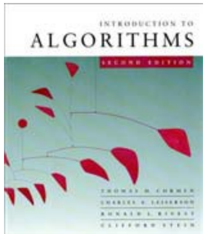
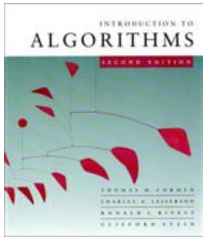
Note: $\delta(u, v) = \infty$ if no path from u to v exists.



Optimal substructure

Theorem. A subpath of a shortest path is a shortest path.



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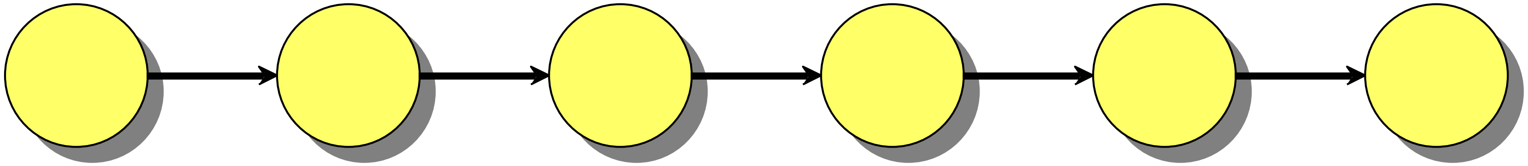
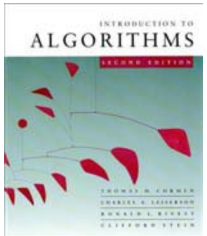
Theorem. A subpath of a shortest path is a shortest path.

Proof. Cut and paste:





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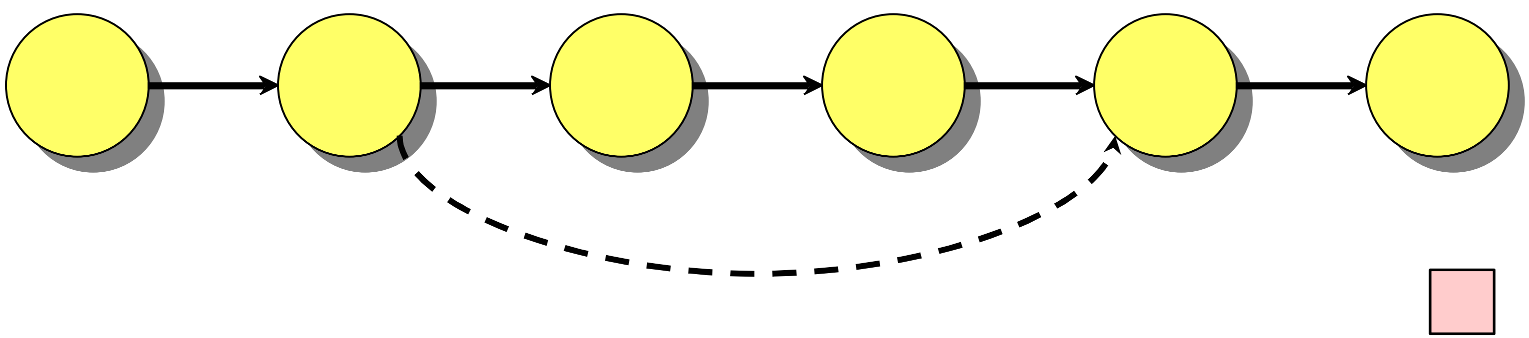
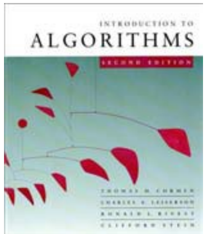
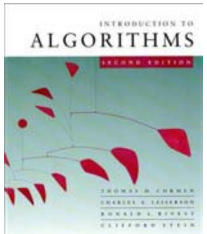




Triangle inequality

Theorem. For all $u, v, x \in V$, we have

$$\delta(u, v) \le \delta(u, x) + \delta(x, v).$$

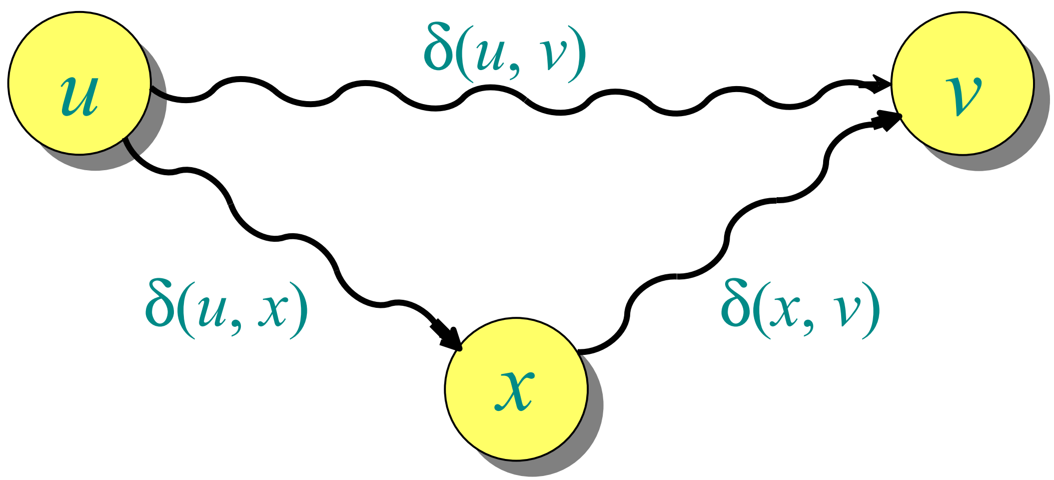


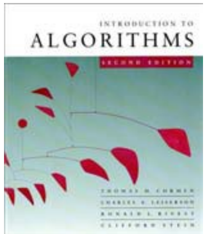
Triangle inequality

Theorem. For all $u, v, x \in V$, we have

$$\delta(u, v) \le \delta(u, x) + \delta(x, v).$$

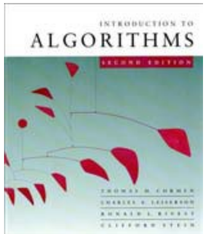
Proof. Assume $\delta(u, v) > \delta(u, x) + \delta(x, v)$





Well-definedness of shortest paths

If a graph *G* contains a negative-weight cycle, then some shortest paths may not exist.

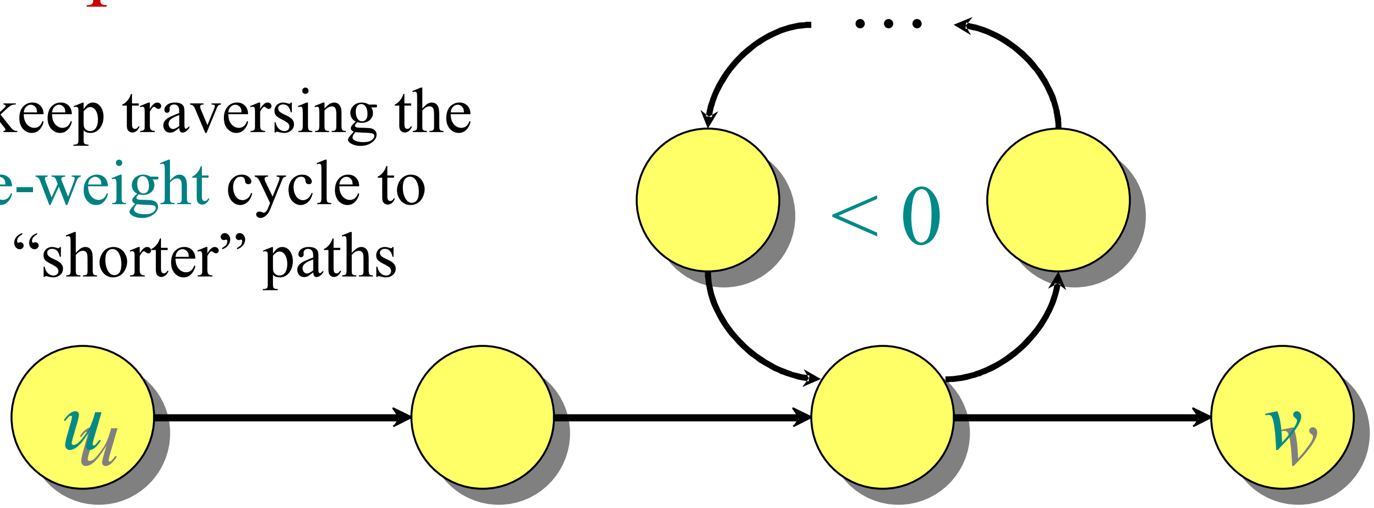


Well-definedness of shortest paths

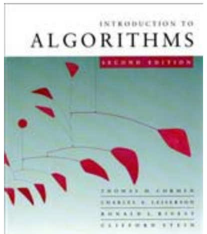
If a graph *G* contains a negative-weight cycle, then some shortest paths may not exist.

Example:

We can keep traversing the negative-weight cycle to get even "shorter" paths



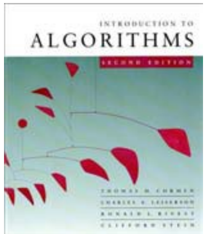
The shortest path weight $\delta(u,v)$ in that case is $-\infty$



Single-source shortest paths

Problem. From a given source vertex $s \in V$, find the shortest-path weights $\delta(s, v)$ for all $v \in V$.

If all edge weights $w(u, v)$ are *nonnegative*, all shortest-path weights must exist.



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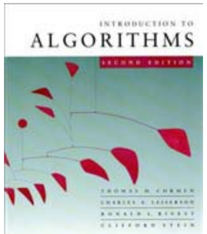
If all edge weights $w(u, v)$ are *nonnegative*, all shortest-path weights must exist.

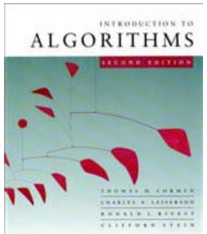
IDEA: Greedy.

1. Maintain a set S of vertices whose shortest-path distances from s are known.
2. At each step add to S the vertex $v \in V - S$ whose distance estimate from s is minimal.
3. Update the distance estimates of vertices adjacent to v.



Dijkstra's algorithm

$d[s] \leftarrow 0$
for each $v \in V - \{s\}$
    do $d[v] \leftarrow \infty$
$S \leftarrow \varnothing$
$Q \leftarrow V$ $\triangleleft$ Q is a priority queue maintaining $V - S$



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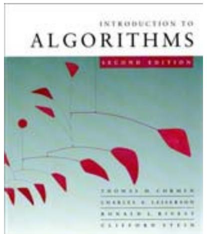
while $Q \neq \varnothing$
    do $u \leftarrow$ EXTRACT-MIN(Q)
        $S \leftarrow S \cup \{u\}$
        for each $v \in Adj[u]$
            do if $d[v] > d[u] + w(u, v)$
                then $d[v] \leftarrow d[u] + w(u, v)$



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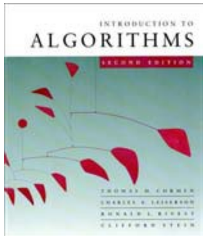
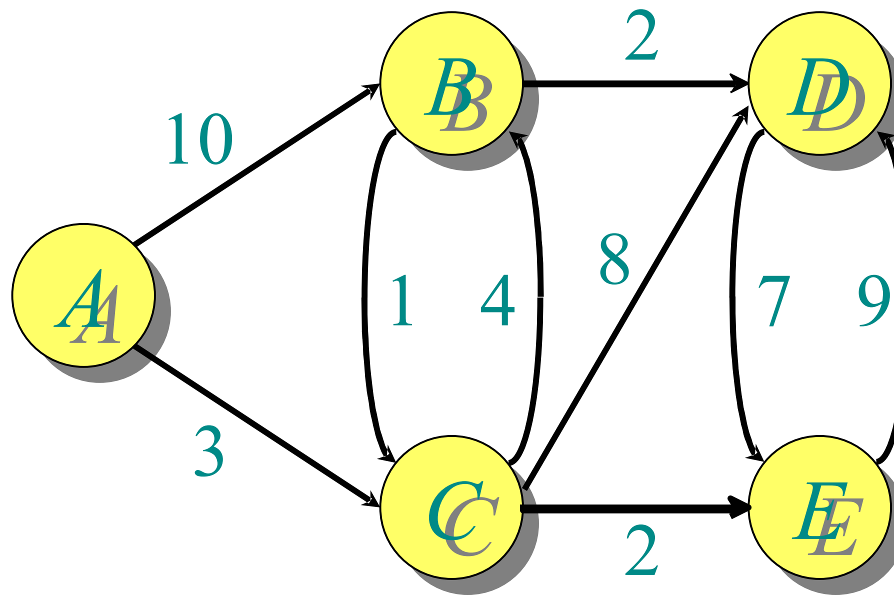
relaxation step

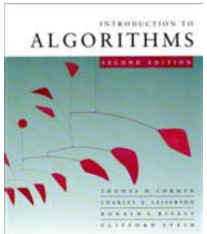
Implicit DECREASE-KEY



Example of Dijkstra's algorithm

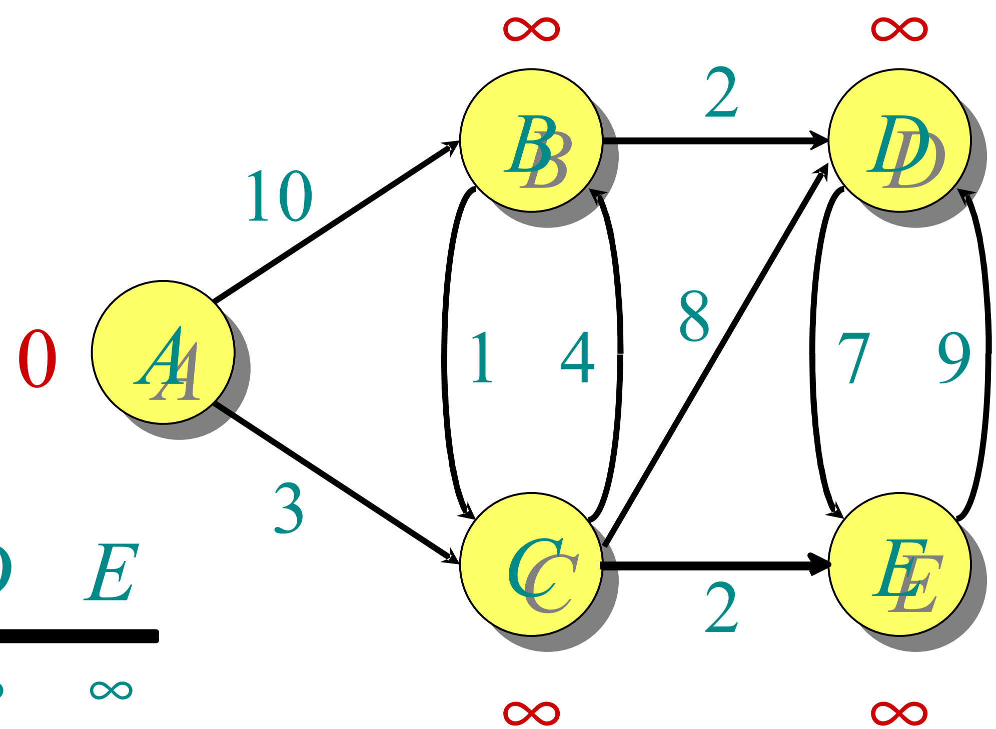
Graph with nonnegative edge weights:





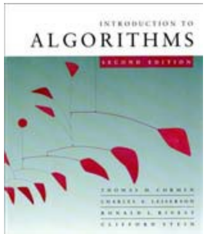
Example of Dijkstra's algorithm

Initialize:



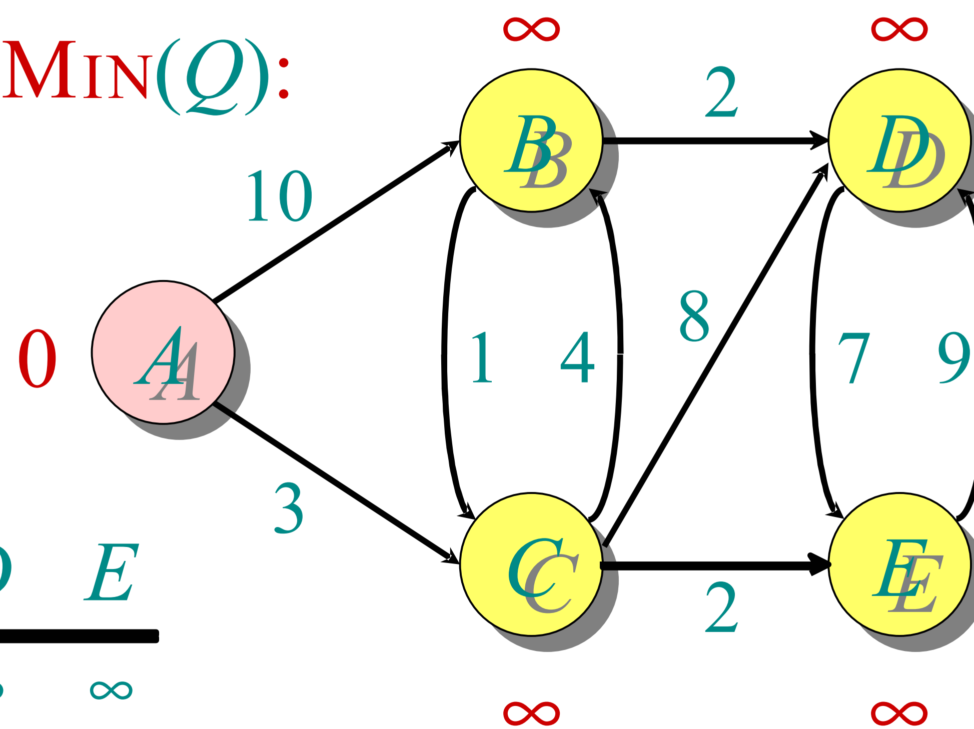
Q:	A	B	C	D	E
	0	∞	∞	∞	∞

S: {}



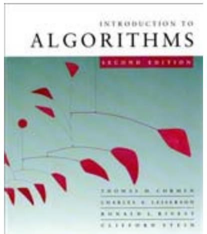
Example of Dijkstra's algorithm

"*A*" ⟸ EXTRACT-MIN(*Q*):



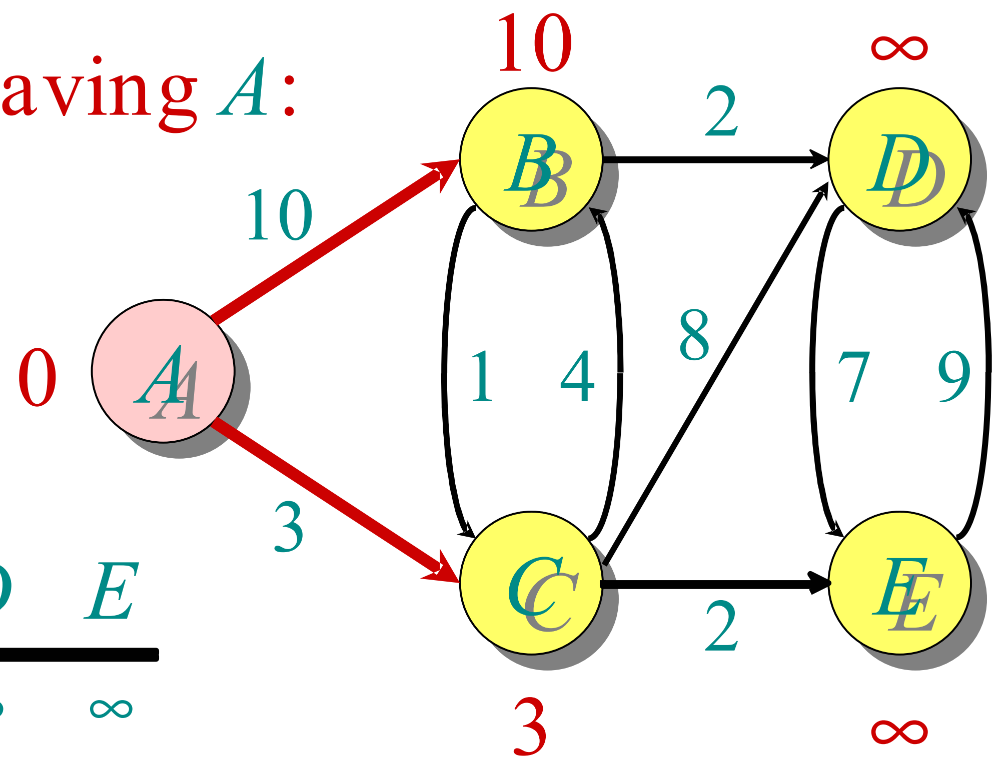
Q:	*A*	*B*	*C*	*D*	*E*
	0	∞	∞	∞	∞

S: { *A* }



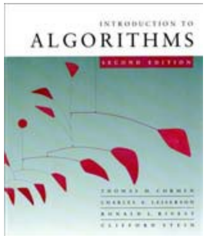
Example of Dijkstra's algorithm

Relax all edges leaving *A*:



Q:	*A*	*B*	*C*	*D*	*E*
	0	∞	∞	∞	∞
		10	3	∞	∞

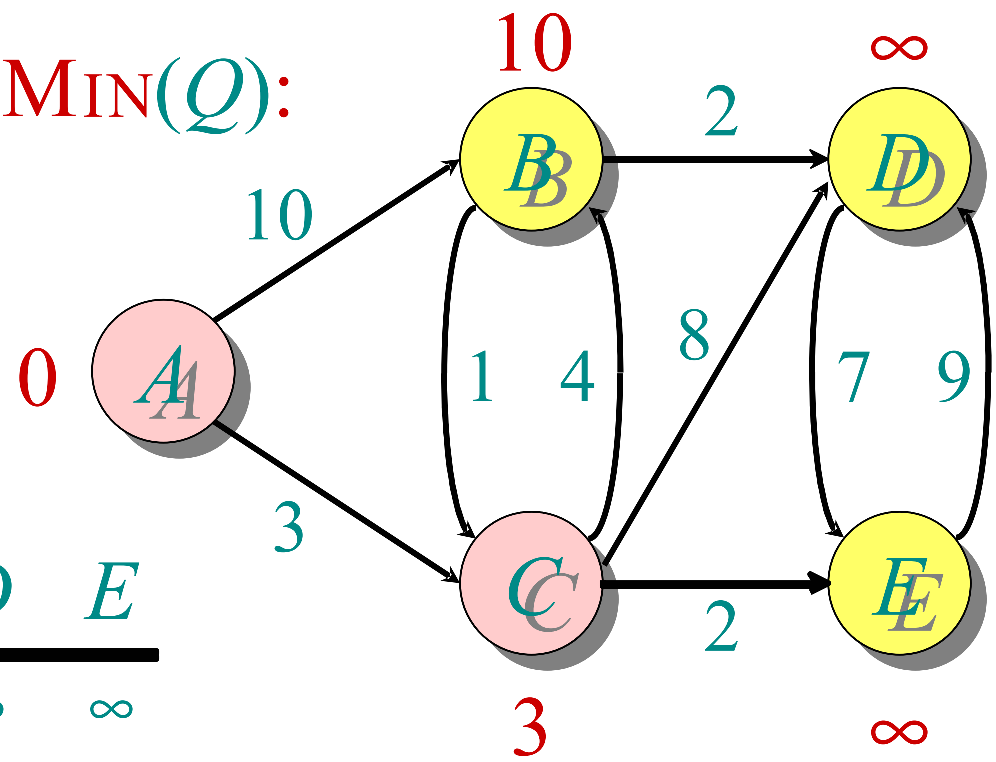
S: { *A* }



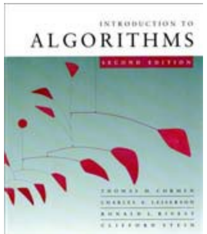
Example of Dijkstra's algorithm

"C" ⟸ EXTRACT-MIN(*Q*):

Q:	*A*	*B*	*C*	*D*	*E*
	0	∞	∞	∞	∞
		10	3	∞	∞

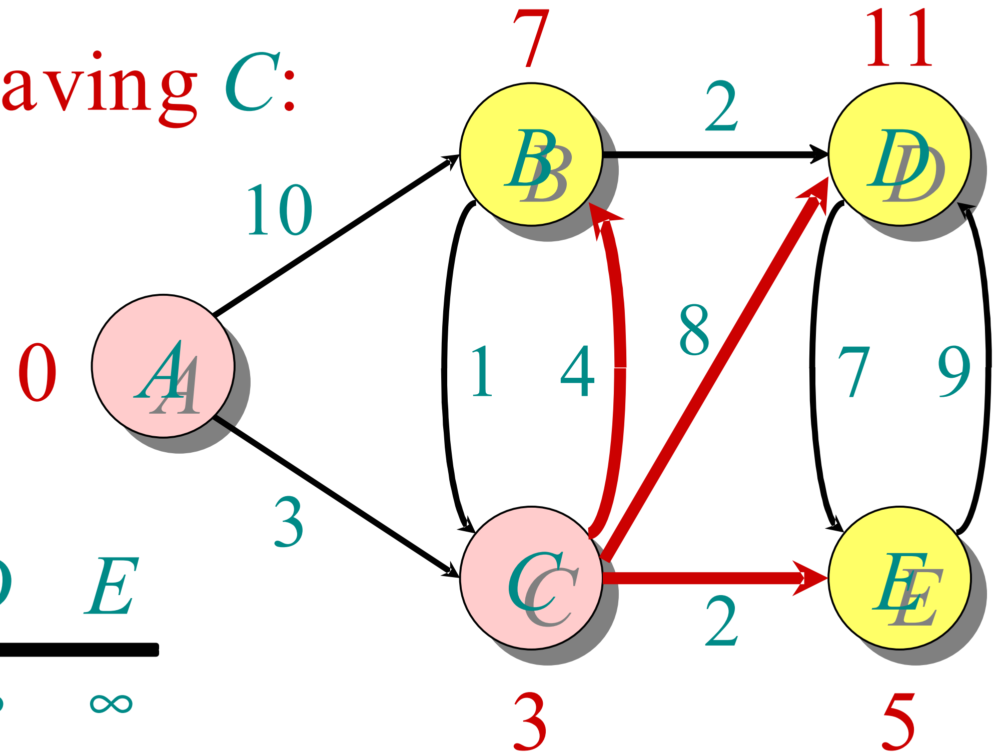


S: { *A*, *C* }



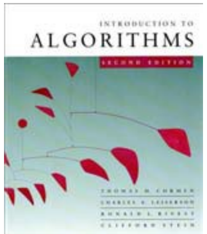
Example of Dijkstra's algorithm

Relax all edges leaving *C*:



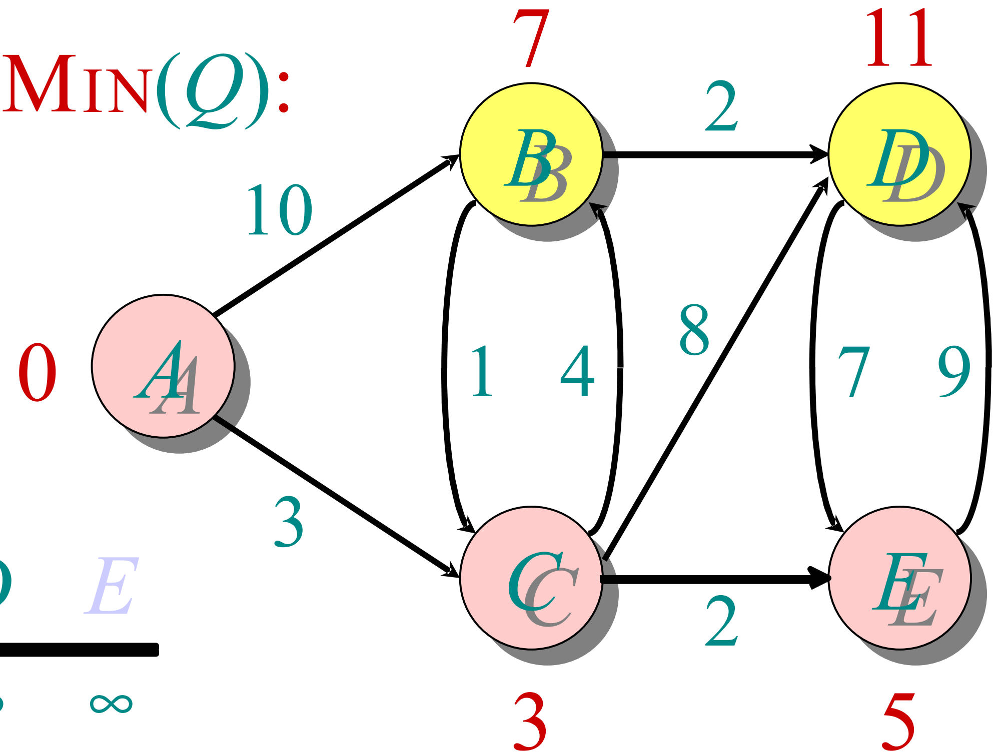
Q:	*A*	*B*	*C*	*D*	*E*
	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5

S: { *A*, *C* }



Example of Dijkstra's algorithm

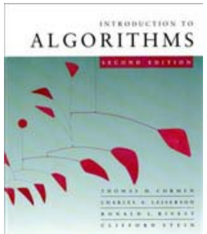
"*E*" ⟸ EXTRACT-MIN(*Q*):



Q:

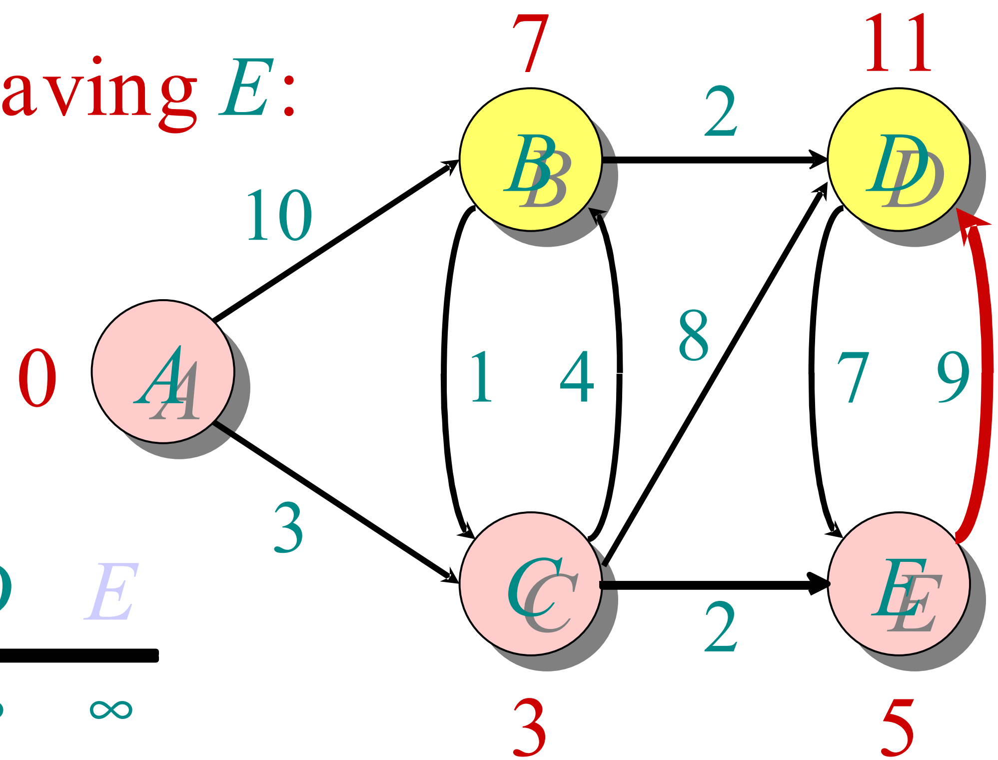
A	*B*	*C*	*D*	*E*
0	∞	∞	∞	∞
	10	3	∞	∞
	7		11	5

S: { *A*, *C*, *E* }



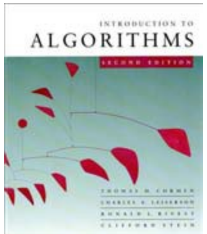
Example of Dijkstra's algorithm

Relax all edges leaving *E*:



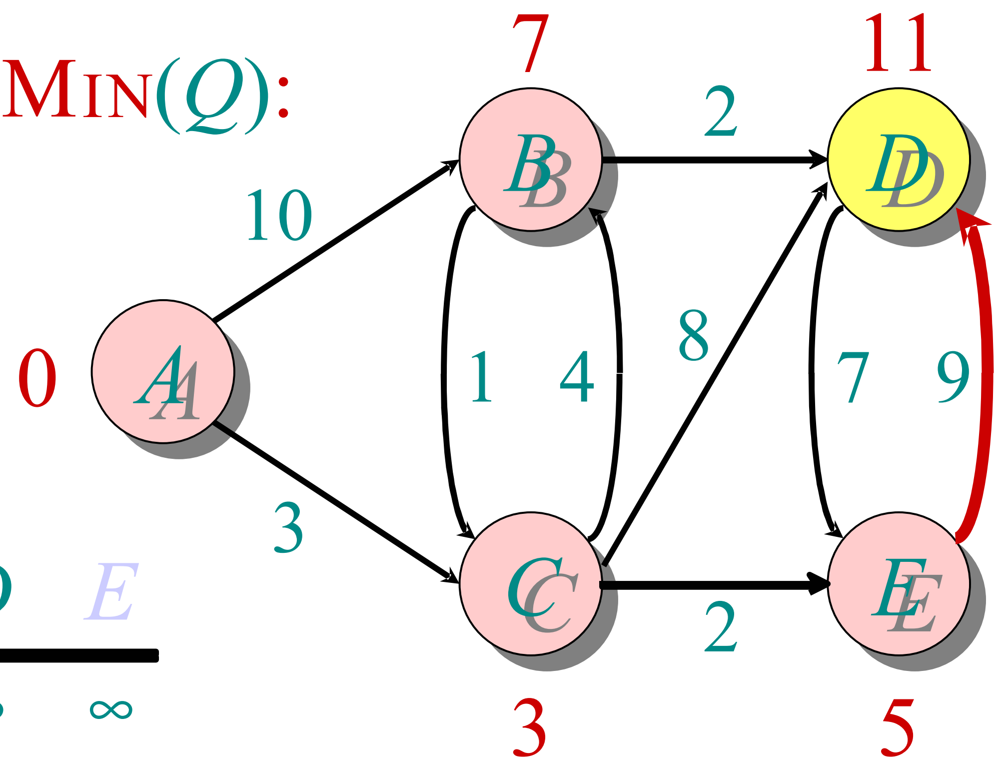
Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5
		7		11	

S: { A, C, E }



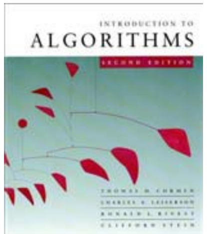
Example of Dijkstra's algorithm

"*B*" ⟸ EXTRACT-MIN(*Q*):



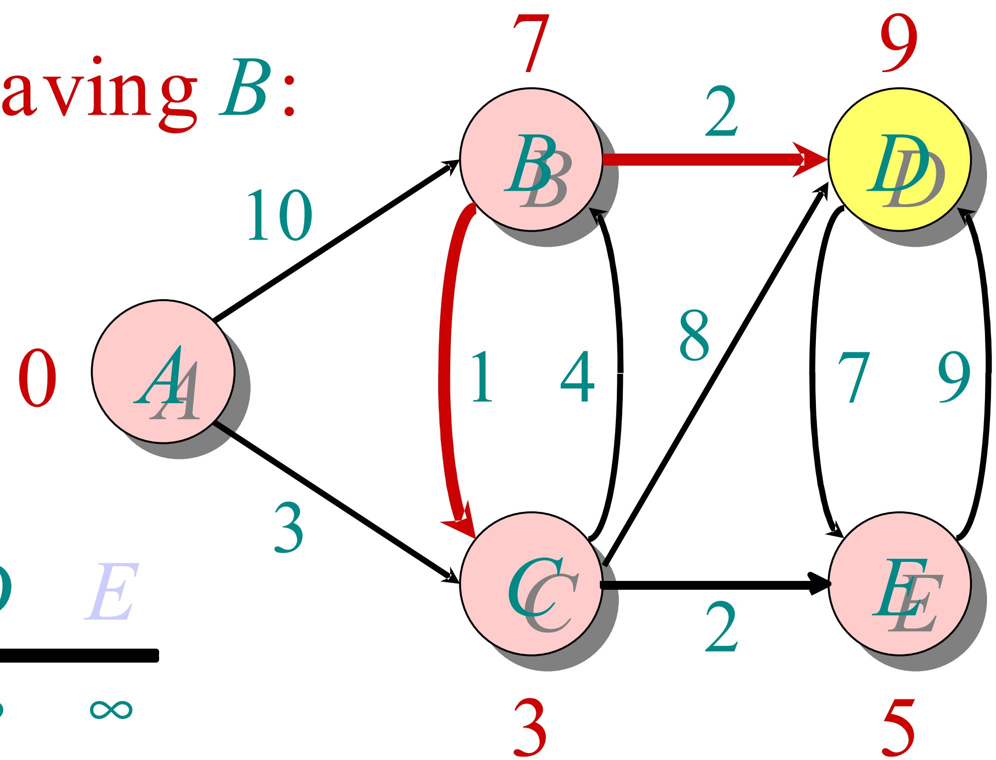
Q:	*A*	*B*	*C*	*D*	*E*
	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5
		7		11	

S: { *A*, *C*, *E*, *B* }



Example of Dijkstra's algorithm

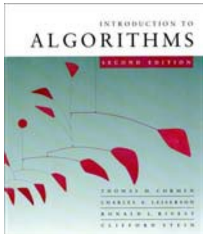
Relax all edges leaving *B*:



Q:

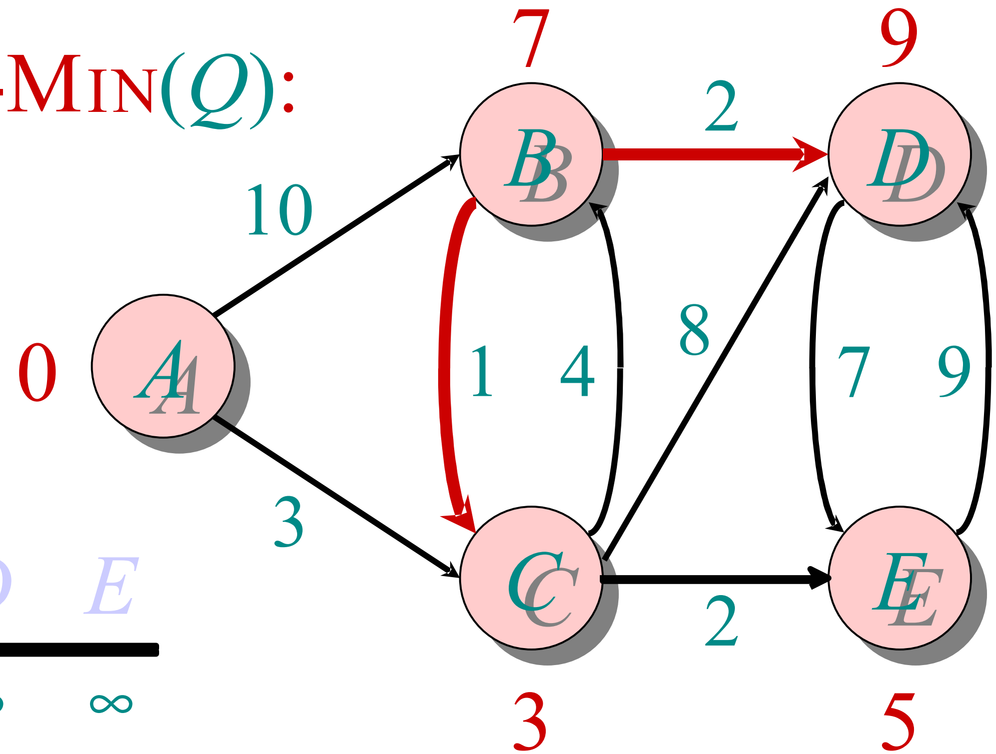
A	*B*	*C*	*D*	*E*
0	∞	∞	∞	∞
	10	3	∞	∞
	7		11	5
	7		11	
			9	

S: { *A*, *C*, *E*, *B* }



Example of Dijkstra's algorithm

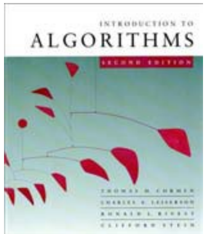
"D" ⟸ EXTRACT-MIN(*Q*):



Q:

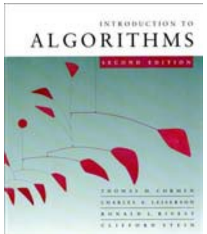
A	*B*	*C*	*D*	*E*
0	∞	∞	∞	∞
	10	3	∞	∞
	7		11	5
	7		11	
			9	

S: { *A*, *C*, *E*, *B*, *D* }



Correctness — Part I

Lemma. Initializing $d[s] \leftarrow 0$ and $d[v] \leftarrow \infty$ for all $v \in V - \{s\}$ establishes $d[v] \geq \delta(s, v)$ for all $v \in V$, and this invariant is maintained over any sequence of relaxation steps.



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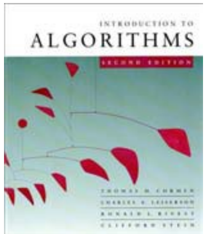
Proof. Suppose not. Let v be the first vertex for which $d[v] < \delta(s, v)$, and let u be the vertex that caused $d[v]$ to change: $d[v] = d[u] + w(u, v)$. Then,



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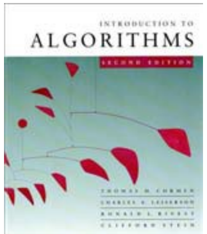
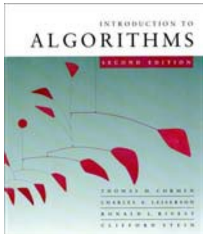
$d[v] < \delta(s, v)$	supposition
$\leq \delta(s, u) + \delta(u, v)$	triangle inequality
$\leq \delta(s,u) + w(u, v)$	sh. path $\leq$ specific path
$\leq d[u] + w(u, v)$	v is first violation

Contradiction. □



Correctness — Part II

Lemma. Let u be v's predecessor on a shortest path from s to v. Then, if $d[u] = \delta(s, u)$ and edge (u, v) is relaxed, we have $d[v] = \delta(s, v)$ after the relaxation.



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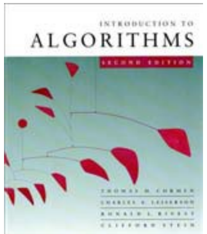
Proof. Observe that $\delta(s, v) = \delta(s, u) + w(u, v)$.

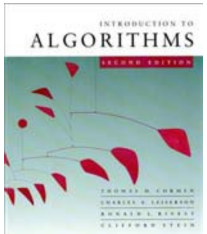
Because u is v's predecessor on the shortest path



Correctness — Part II

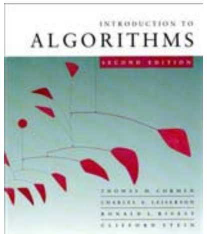
Lemma. Let u be v's predecessor on a shortest path from s to v. Then, if $d[u] = \delta(s, u)$ and edge (u, v) is relaxed, we have $d[v] = \delta(s, v)$ after the relaxation.

Proof. Observe that $\delta(s, v) = \delta(s, u) + w(u, v)$. Suppose that $d[v] > \delta(s, v)$ before the relaxation. (Otherwise, we're done.) Then, the test $d[v] > d[u] + w(u, v)$ succeeds, because $d[v] > \delta(s, v) = \delta(s, u) + w(u, v) = d[u] + w(u, v)$, and the algorithm sets $d[v] = d[u] + w(u, v) = \delta(s, v)$. ∎



Correctness — Part III

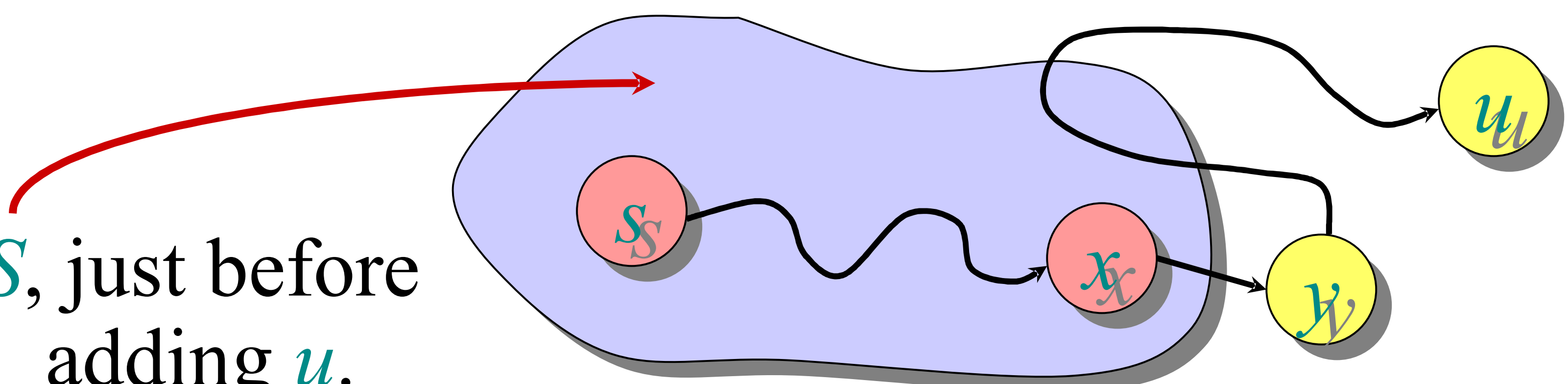
Theorem. Dijkstra's algorithm terminates with $d[v] = \delta(s, v)$ for all $v \in V$.



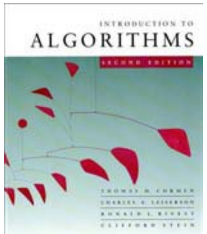
Correctness — Part III

Theorem. Dijkstra's algorithm terminates with $d[v] = \delta(s, v)$ for all $v \in V$.

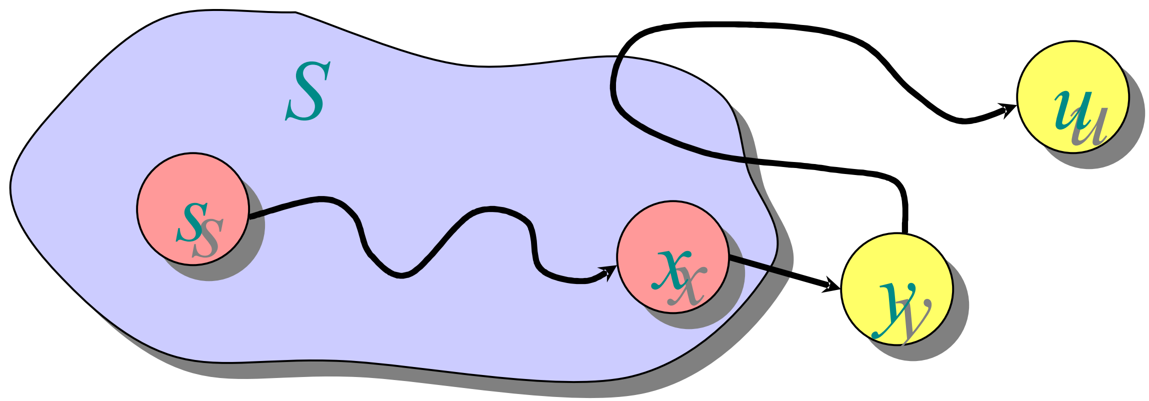
Proof. It suffices to show that $d[v] = \delta(s, v)$ for every $v \in V$ when v is added to S. Suppose u is the first vertex added to S for which $d[u] > \delta(s, u)$. Let y be the first vertex in $V - S$ along a shortest path from s to u, and let x be its predecessor:



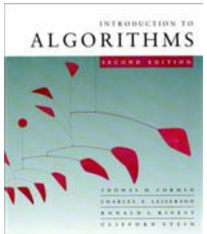
S, just before adding u.



Correctness — Part III (continued)

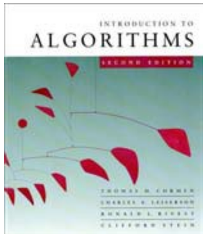


Since u is the first vertex violating the claimed invariant, we have $d[x] = \delta(s, x)$. When x was added to S, the edge (x, y) was relaxed, which implies that $d[y] = \delta(s, y) \le \delta(s, u) < d[u]$. But, $d[u] \le d[y]$ by our choice of u. Contradiction. □



Analysis of Dijkstra

while $Q \neq \emptyset$
  do $u \leftarrow$ EXTRACT-MIN(Q)
    $S \leftarrow S \cup \{u\}$
    for each $v \in Adj[u]$
      do if $d[v] > d[u] + w(u, v)$
        then $d[v] \leftarrow d[u] + w(u, v)$



Analysis of Dijkstra

$|V|$ times:

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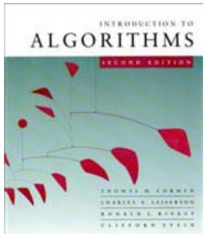
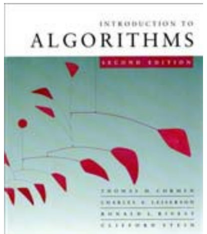


Analysis of Dijkstra

$|V|$ times:

while $Q \neq \emptyset$
  do $u \leftarrow$ EXTRACT-MIN(Q)
    $S \leftarrow S \cup \{u\}$
    $degree(u)$ times:
    for each $v \in Adj[u]$
      do if $d[v] > d[u] + w(u, v)$
        then $d[v] \leftarrow d[u] + w(u, v)$



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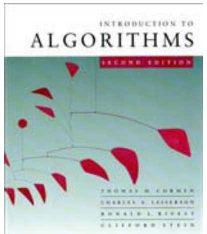
Handshaking Lemma $\Rightarrow \Theta(E)$ implicit DECREASE-KEY's.



Analysis of Dijkstra

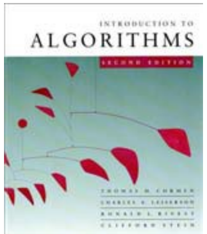
$|V|$ times:

while $Q \neq \emptyset$
  do $u \leftarrow$ EXTRACT-MIN(Q)
    $S \leftarrow S \cup \{u\}$
    for each $v \in Adj[u]$ — *degree*(u) times
      do if $d[v] > d[u] + w(u, v)$
        then $d[v] \leftarrow d[u] + w(u, v)$

Handshaking Lemma $\Rightarrow \Theta(E)$ implicit DECREASE-KEY's.

Time $= \Theta(V \cdot T_{\text{EXTRACT-MIN}} + E \cdot T_{\text{DECREASE-KEY}})$

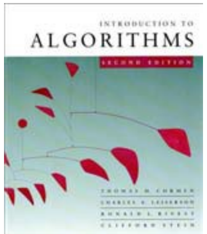
Note: Same formula as in the analysis of Prim's minimum spanning tree algorithm.



Analysis of Dijkstra (continued)

Time = $\Theta(V) \cdot T_{\text{EXTRACT-MIN}} + \Theta(E) \cdot T_{\text{DECREASE-KEY}}$

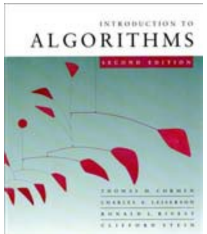
Q	$T_{\text{EXTRACT-MIN}}$	$T_{\text{DECREASE-KEY}}$	Total



Analysis of Dijkstra (continued)

Time = $\Theta(V)\cdot T_{\text{EXTRACT-MIN}} + \Theta(E)\cdot T_{\text{DECREASE-KEY}}$

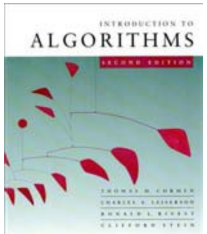
Q	$T_{\text{EXTRACT-MIN}}$	$T_{\text{DECREASE-KEY}}$	Total
array	$O(V)$	$O(1)$	$O(V^2)$



Analysis of Dijkstra (continued)

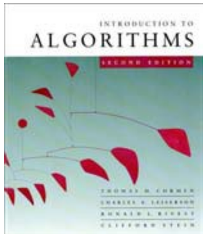
Time = $\Theta(V)\cdot T_{\text{EXTRACT-MIN}} + \Theta(E)\cdot T_{\text{DECREASE-KEY}}$

Q	$T_{\text{EXTRACT-MIN}}$	$T_{\text{DECREASE-KEY}}$	Total
array	$O(V)$	$O(1)$	$O(V^2)$
binary heap	$O(\lg V)$	$O(\lg V)$	$O(E \lg V)$



Unweighted graphs

Suppose that $w(u, v) = 1$ for all $(u, v) \in E$.
Can Dijkstra's algorithm be improved?



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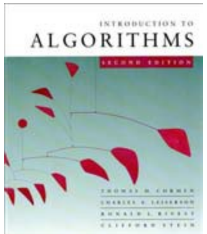
Can Dijkstra's algorithm be improved?

- Use a simple FIFO queue instead of a priority queue.



Unweighted graphs

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Breadth-first search

while $Q \neq \emptyset$
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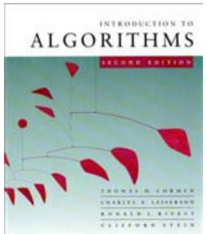
          ENQUEUE(Q, v)



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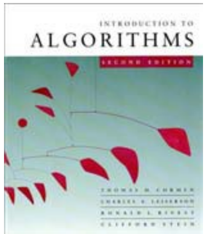
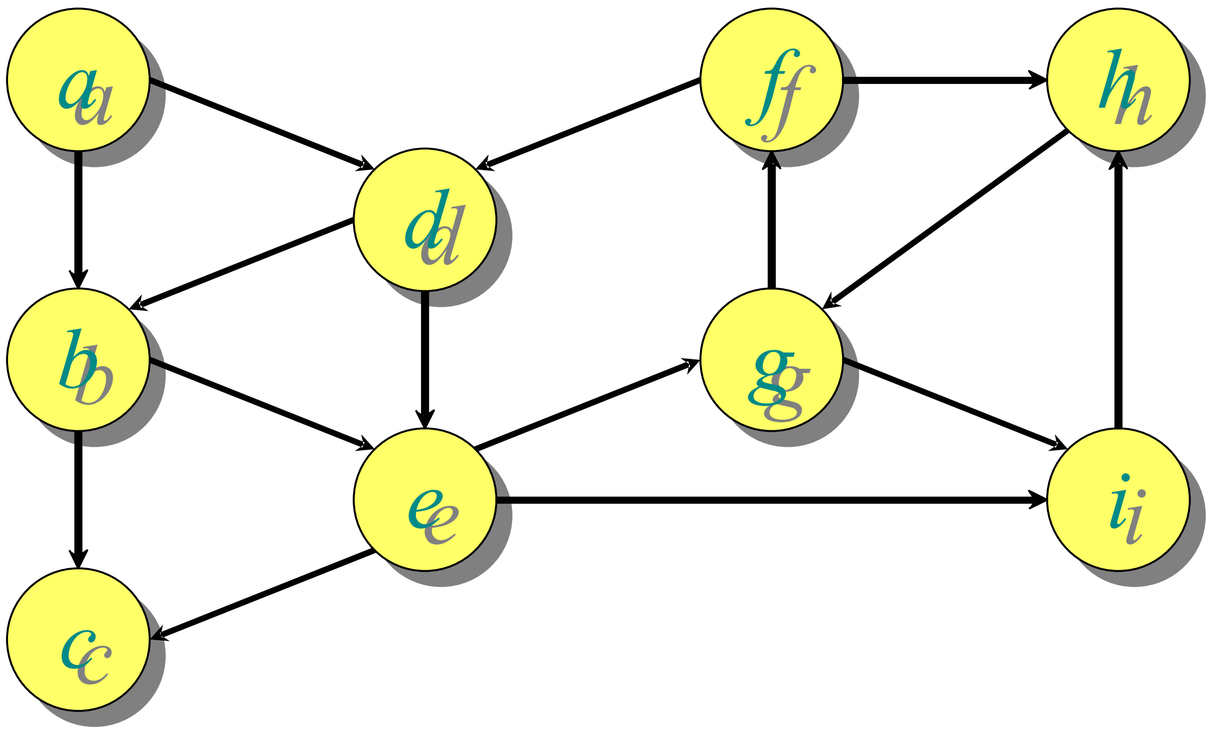
                    ENQUEUE(Q, v)

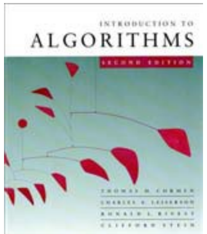
Analysis: Time $= O(V + E)$.



Example of breadth-first search



Q:



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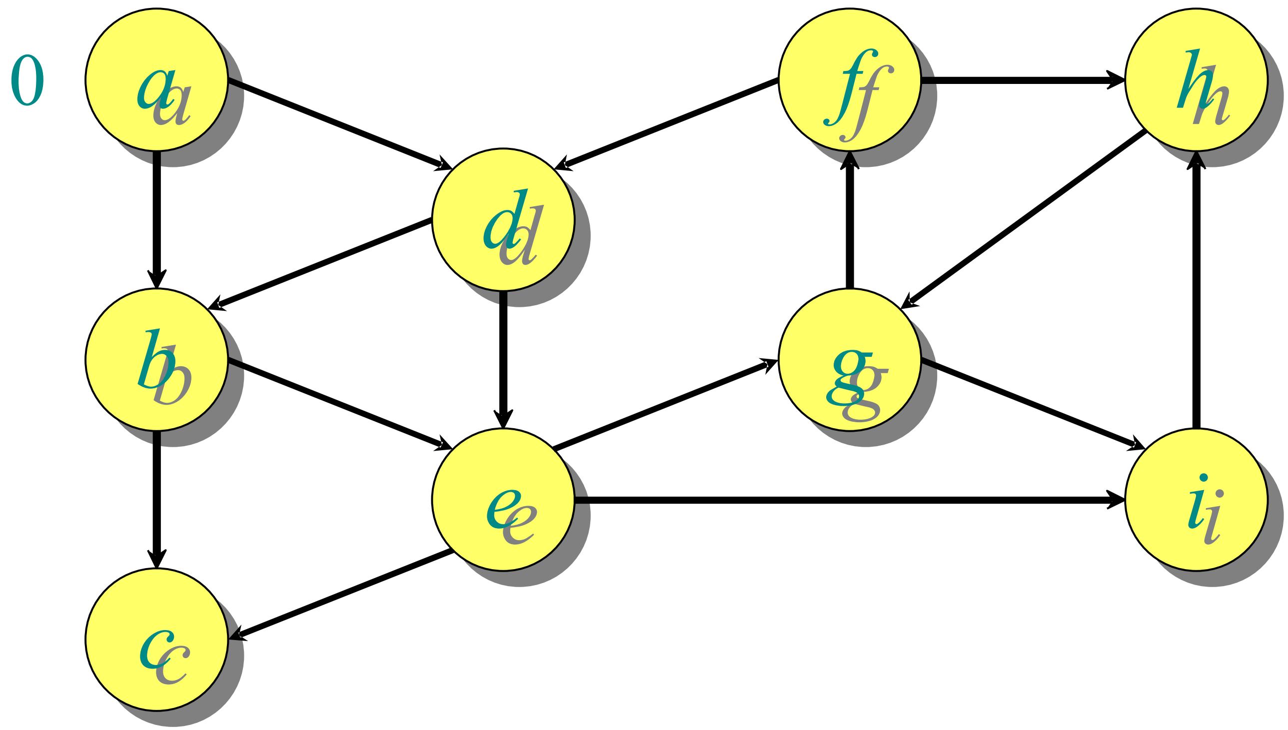
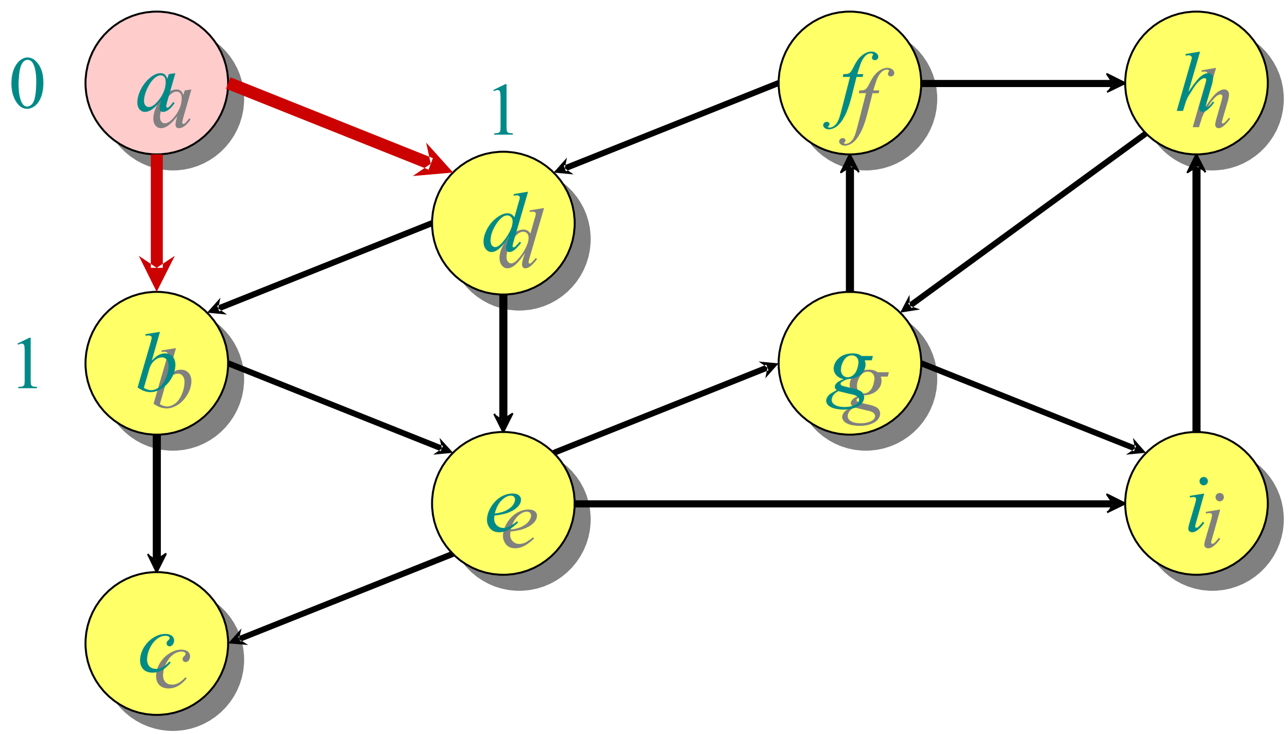


0

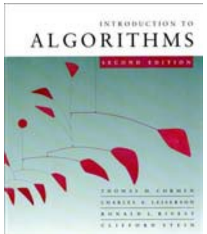
Q: *a*

Example of breadth-first search

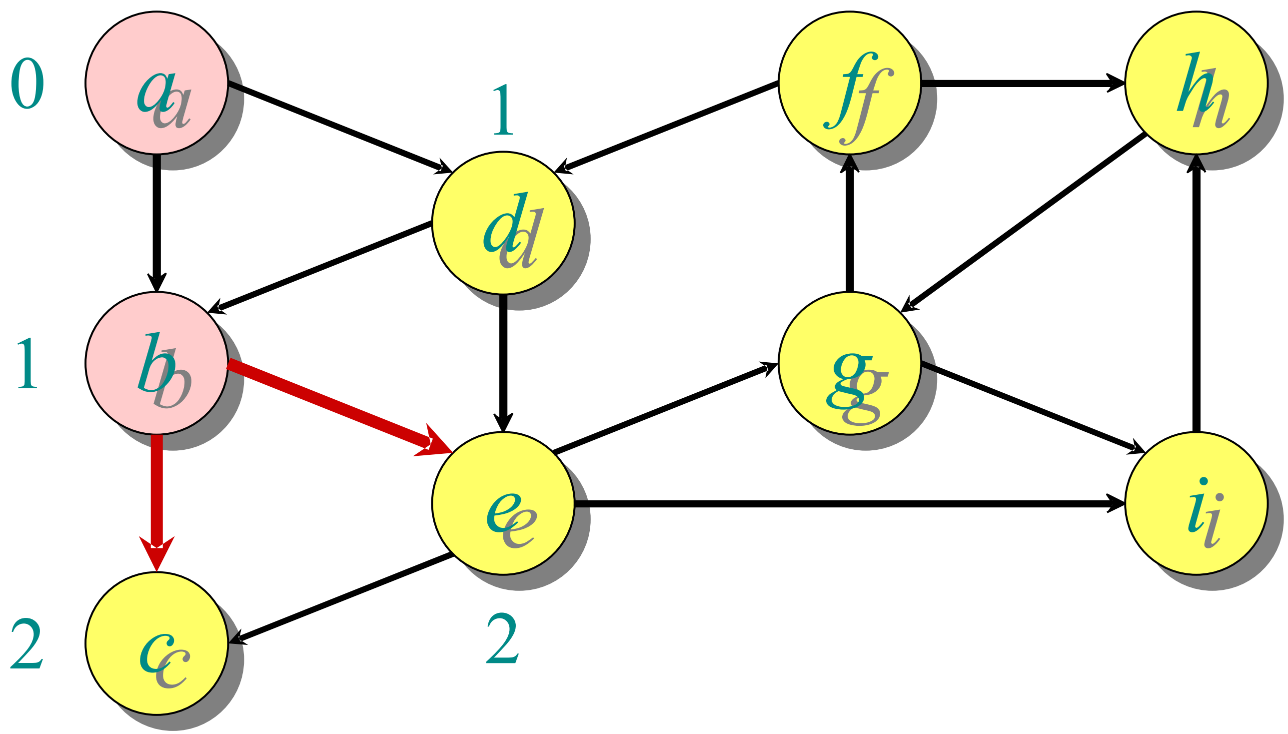


1 1

Q: *a* *b* *d*



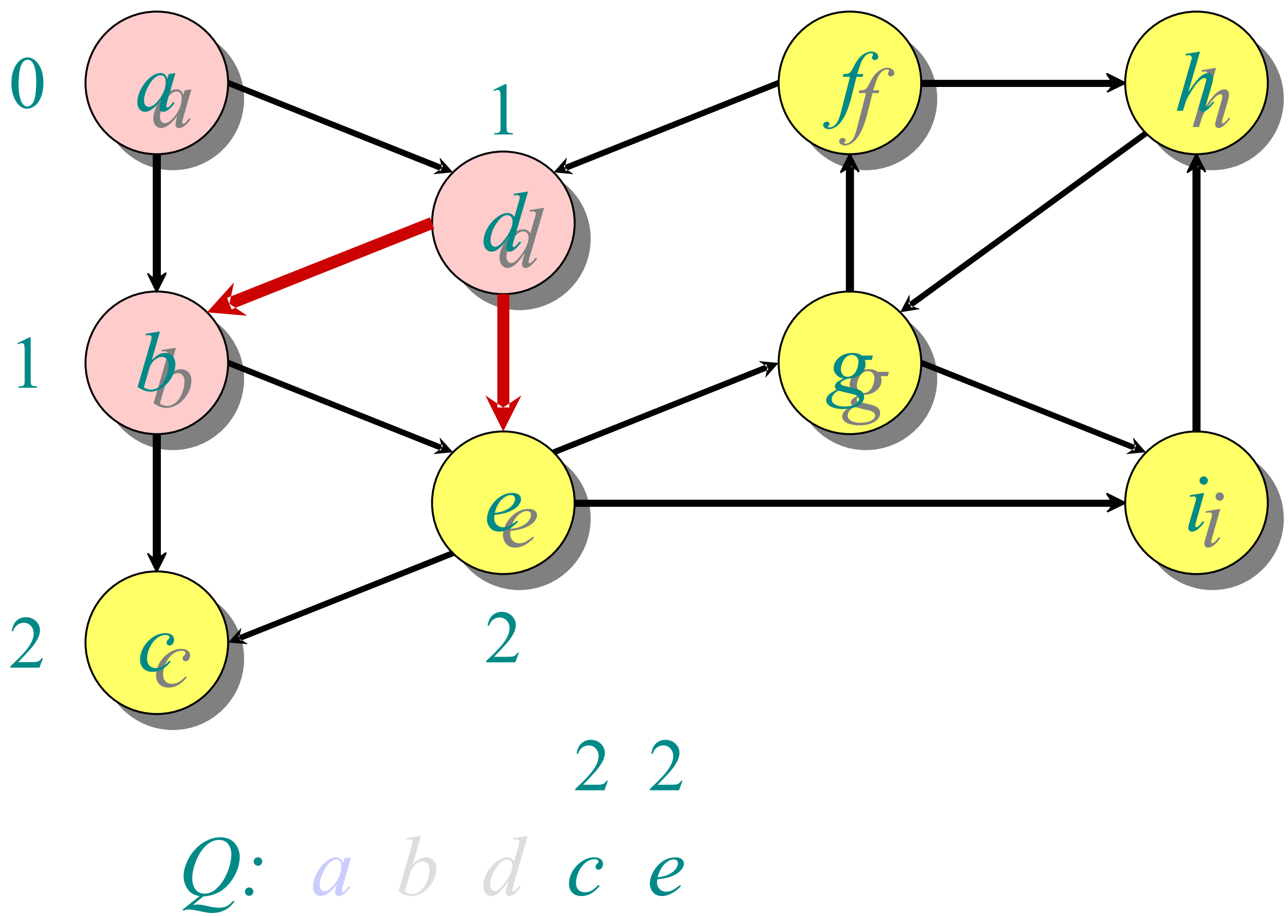
Example of breadth-first search

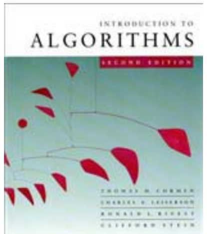


1 2 2

Q: *a* *b* *d* *c* *e*

Example of breadth-first search





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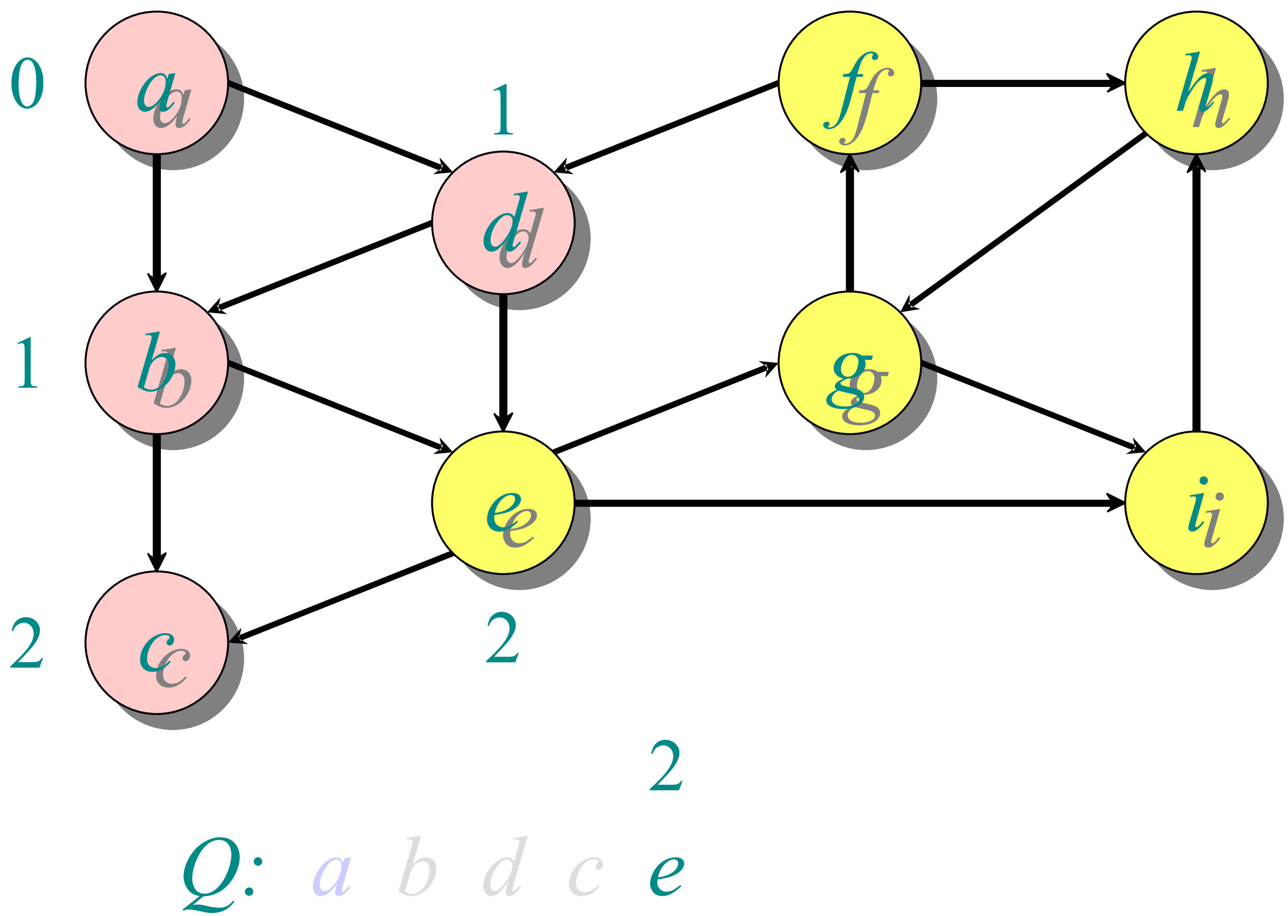

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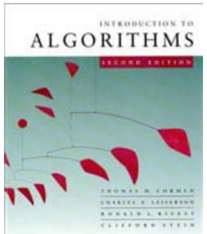

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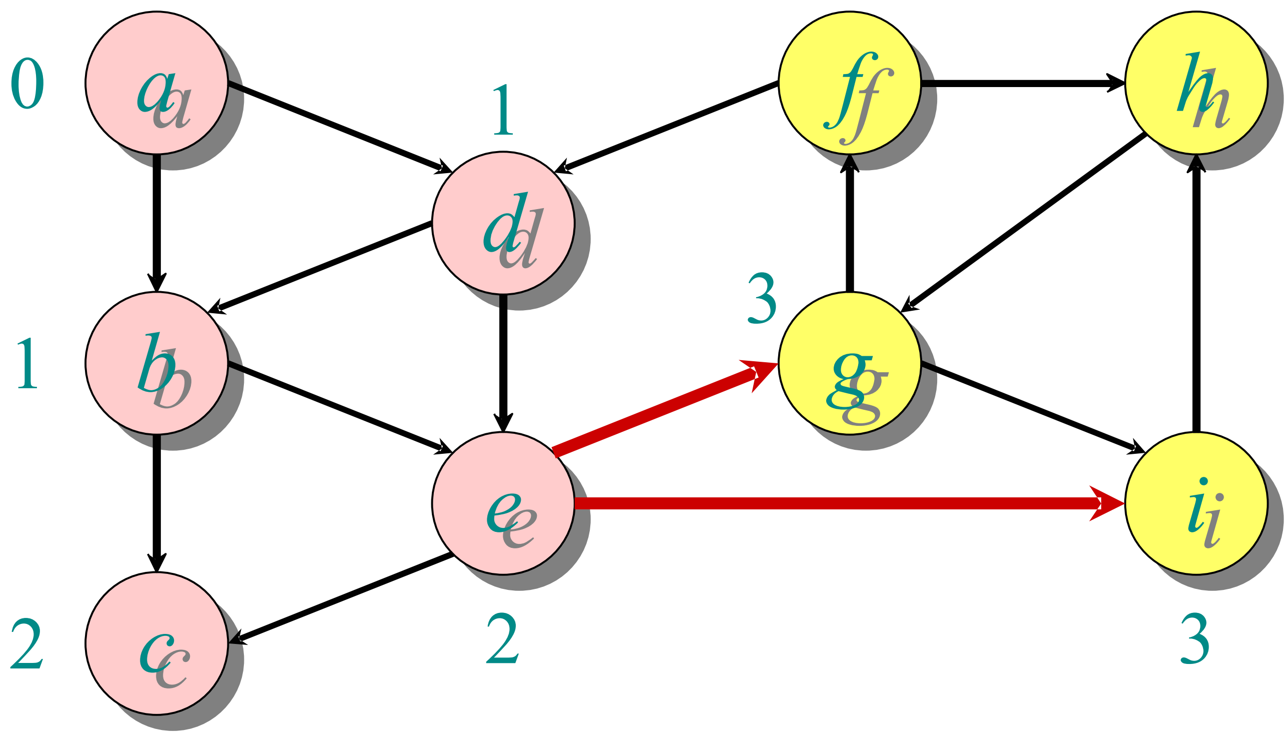

3 3

Q: *a b d c e g i*

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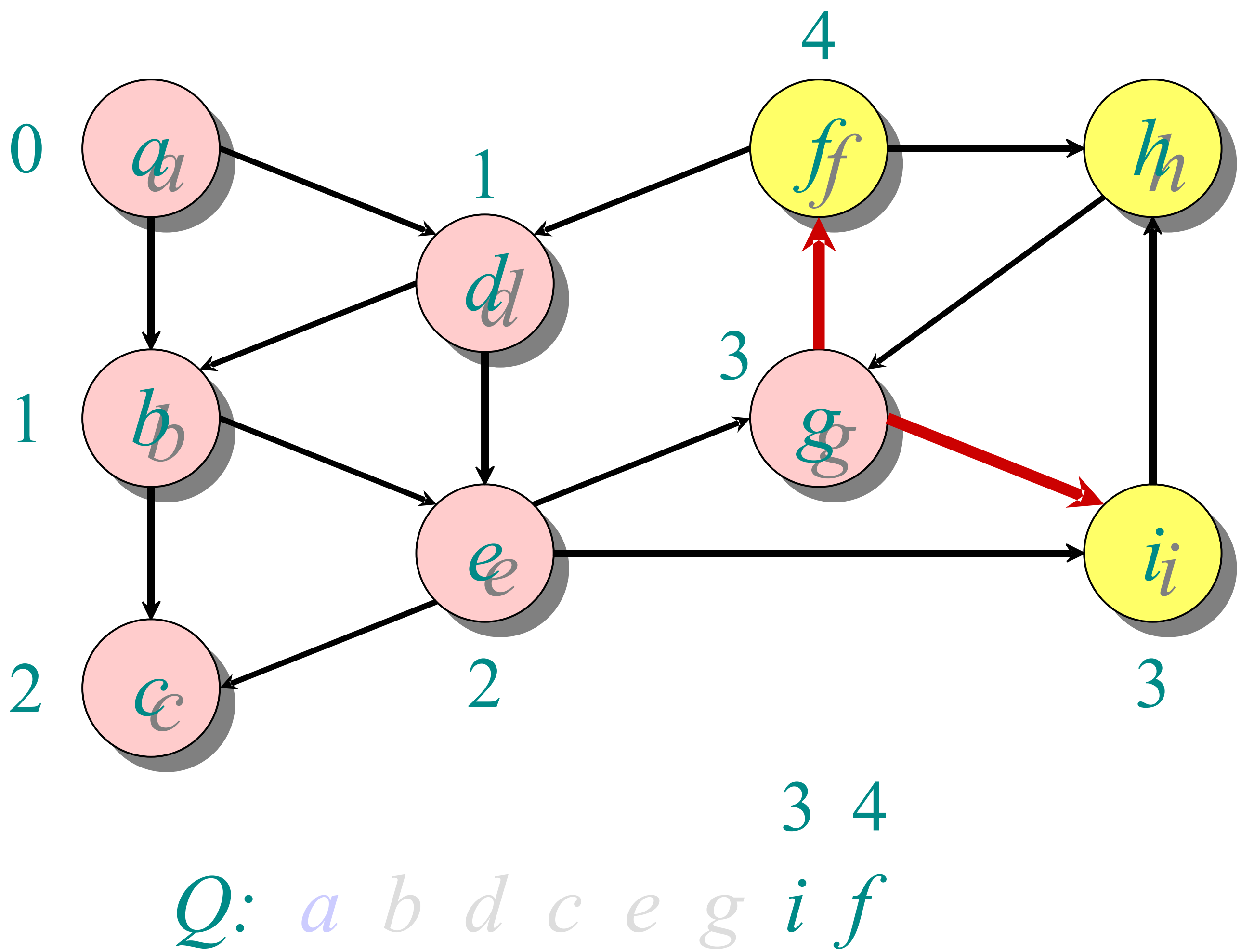
Example of breadth-first search



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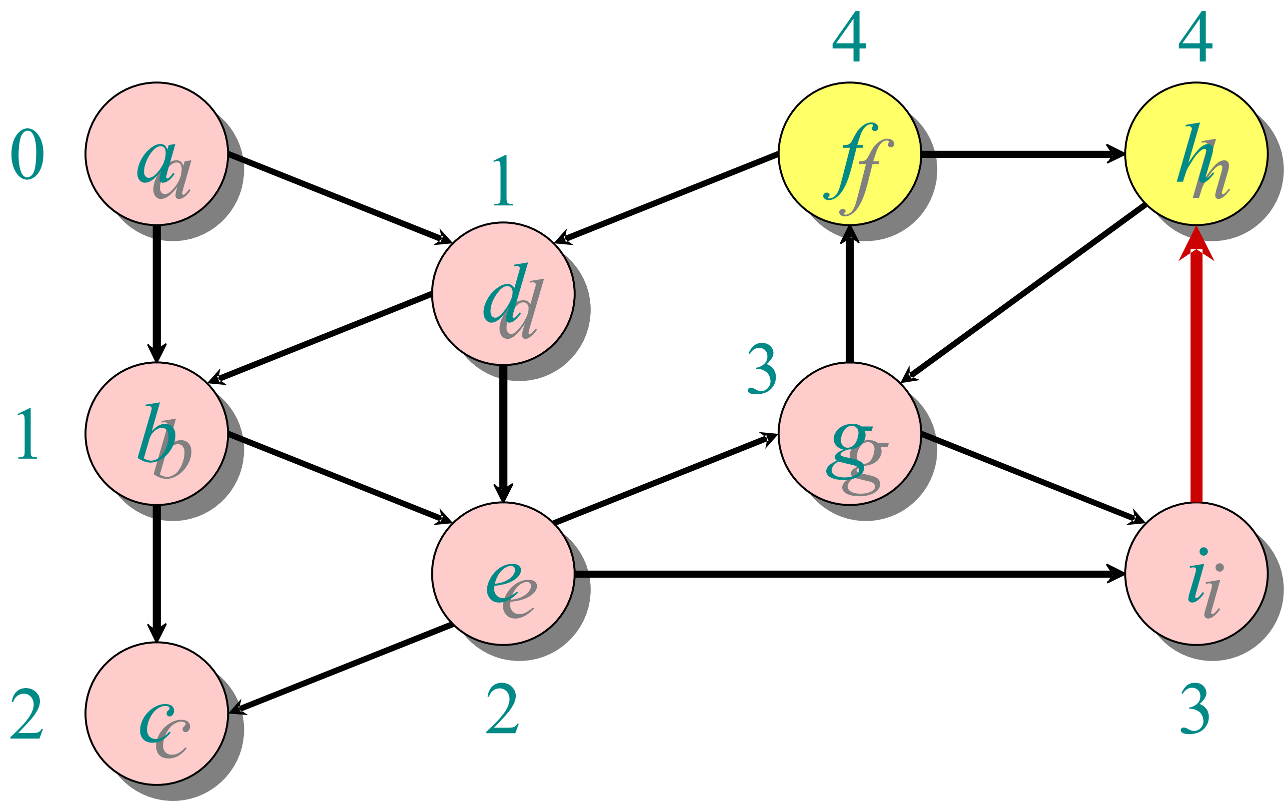
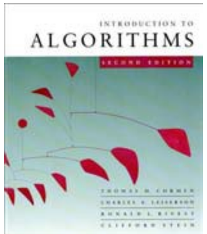


4 4

Q: a b d c e g i f h



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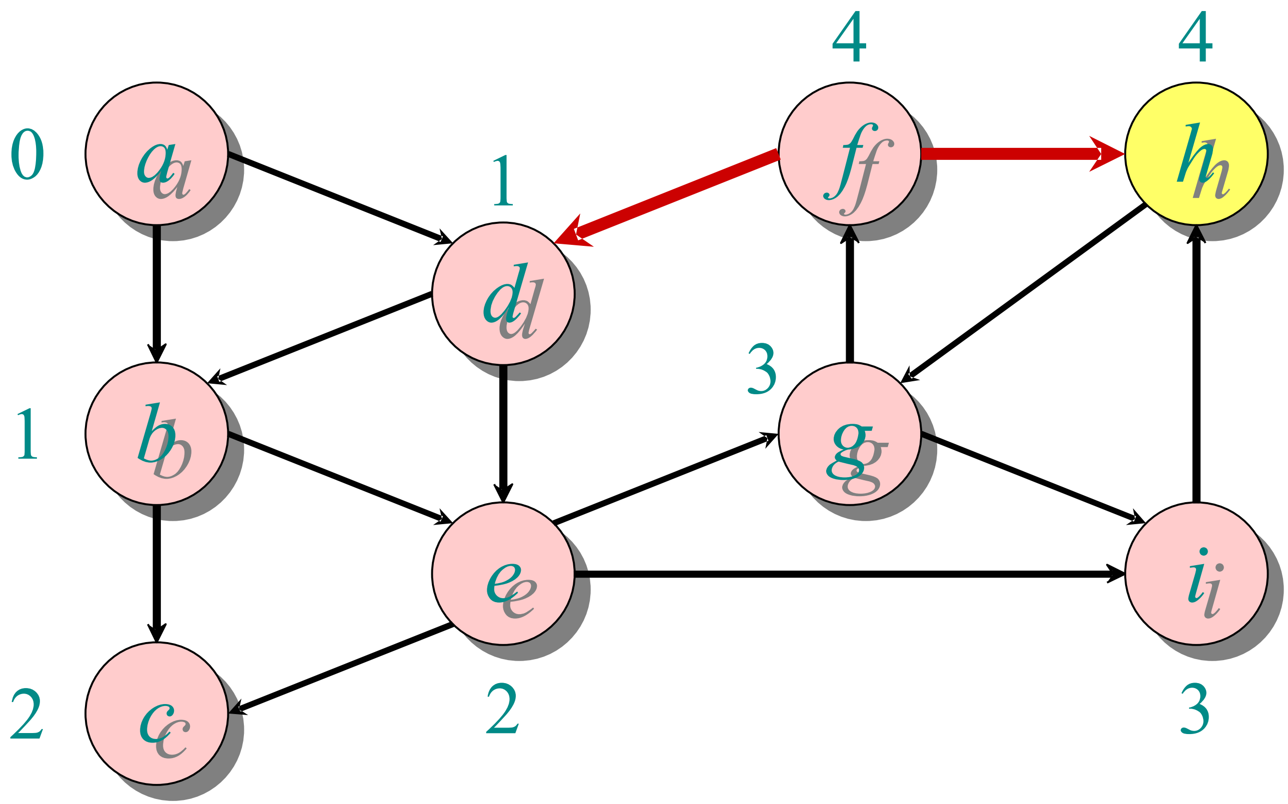
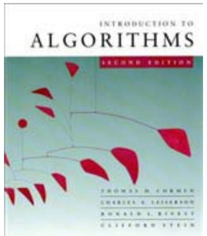


4

Q: *a b d c e g i f h*



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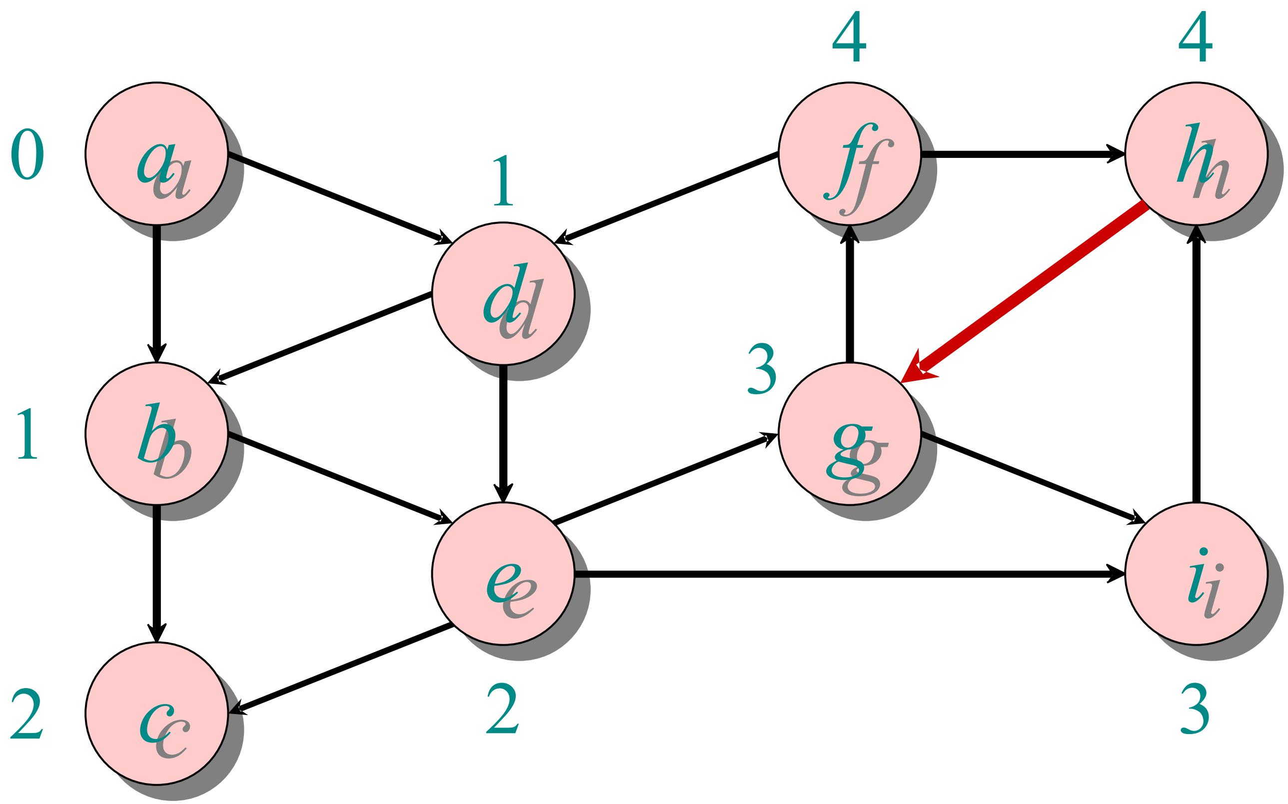
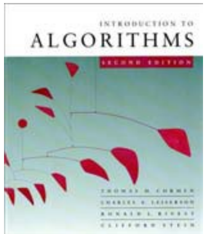


Q: *a b d c e g i f h*



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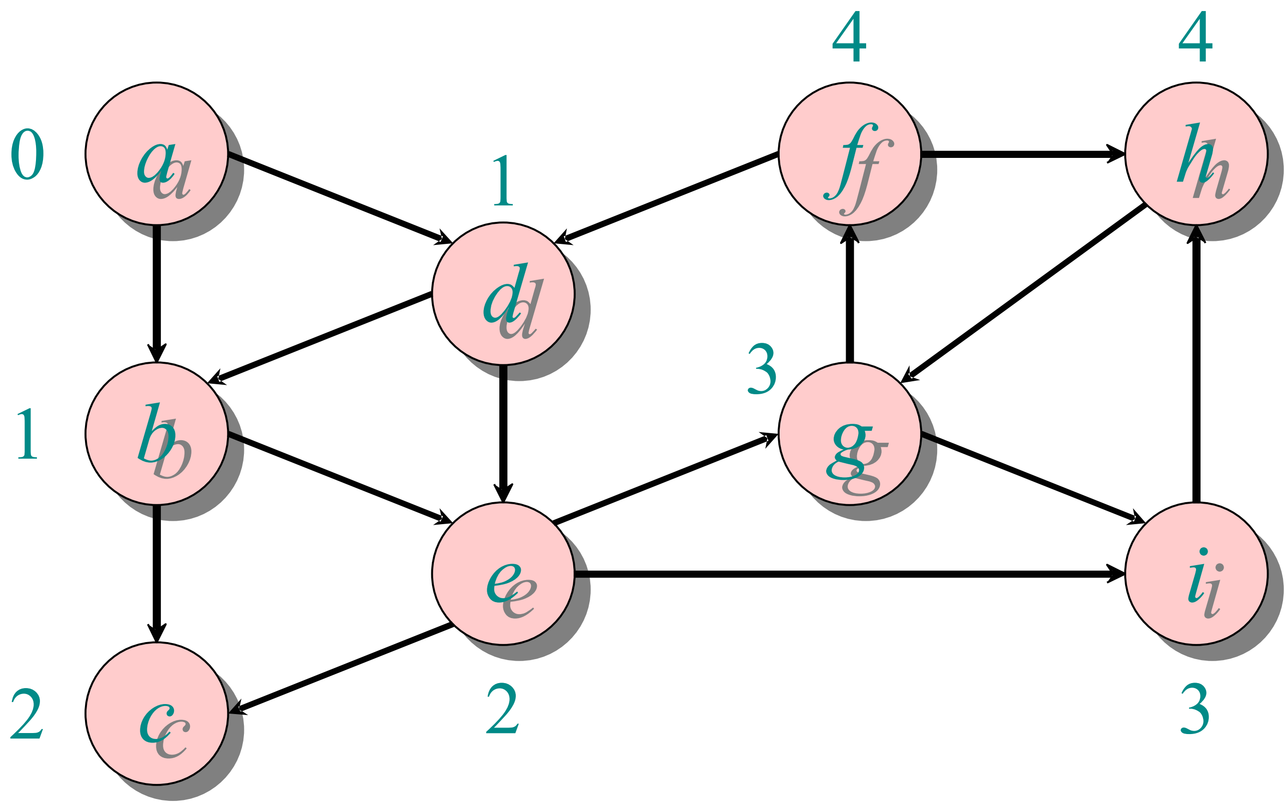
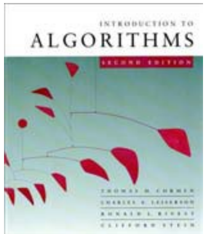


Q: *a b d c e g i f h*



Correctness of BFS

while $Q \neq \emptyset$
  do $u \leftarrow$ DEQUEUE(Q)
    for each $v \in Adj[u]$
      do if $d[v] = \infty$
        then $d[v] \leftarrow d[u] + 1$
          ENQUEUE(Q, v)

Key idea:

The FIFO Q in breadth-first search mimics the priority queue Q in Dijkstra.

- Invariant: v comes after u in Q implies that $d[v] = d[u]$ or $d[v] = d[u] + 1$.

Recap

- Properties of shortest paths
- Dijkstra's Algorithm
- Correctness
- Analysis
- Breadth-First Search