

CMP448: Algorithms



Lecture 16: Bellman-Ford Algorithm

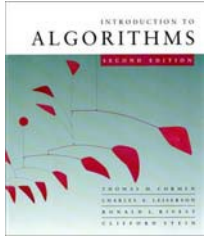
Mohamed Alaa El-Dien Aly
Computer Engineering Department
Cairo University
Fall 2013

Agenda

- Bellman-Ford Algorithm
- DAG Shortest Paths
- Linear Programming and Difference Constraints
- VLSI Layout Compaction

Acknowledgment

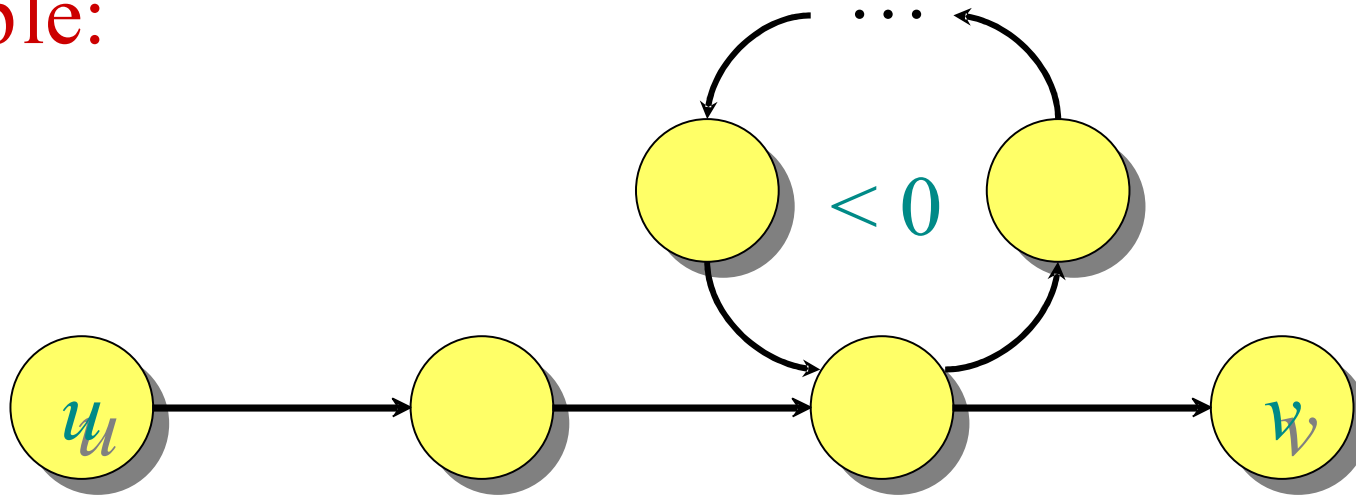
A lot of slides adapted from the slides of Erik Demaine and Charles Leiserson.

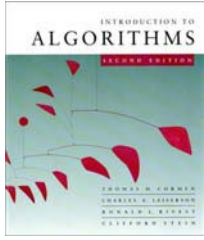


Negative-weight cycles

Recall: If a graph $G = (V, E)$ contains a negative-weight cycle, then some shortest paths may not exist.

Example:

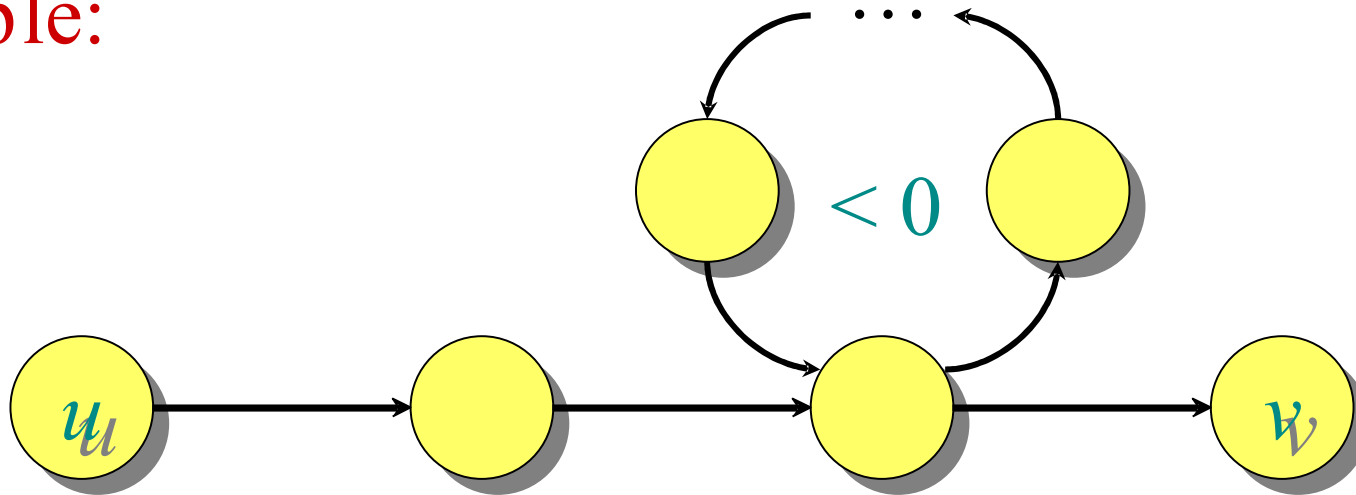




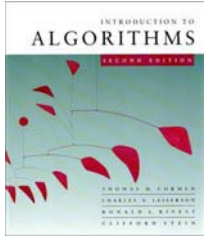
Negative-weight cycles

Recall: If a graph $G = (V, E)$ contains a negative-weight cycle, then some shortest paths may not exist.

Example:



Bellman-Ford algorithm: Finds all shortest-path lengths from a *source* $s \in V$ to all $v \in V$ or determines that a negative-weight cycle exists.



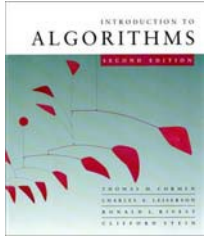
Bellman-Ford algorithm

$d[s] \leftarrow 0$

for each $v \in V - \{s\}$

do $d[v] \leftarrow \infty$

} initialization



Bellman-Ford algorithm

$d[s] \leftarrow 0$

for each $v \in V - \{s\}$

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} initialization

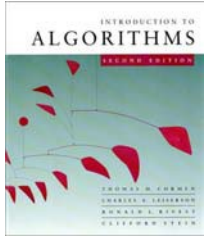
for $i \leftarrow 1$ to $|V| - 1$

do for each edge $(u, v) \in E$

do if $d[v] > d[u] + w(u, v)$

then $d[v] \leftarrow d[u] + w(u, v)$

} *relaxation
step*



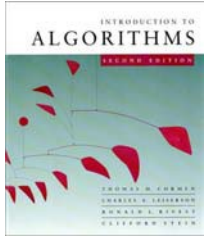
Bellman-Ford algorithm

```
 $d[s] \leftarrow 0$   
for each  $v \in V - \{s\}$   
do  $d[v] \leftarrow \infty$  } initialization
```

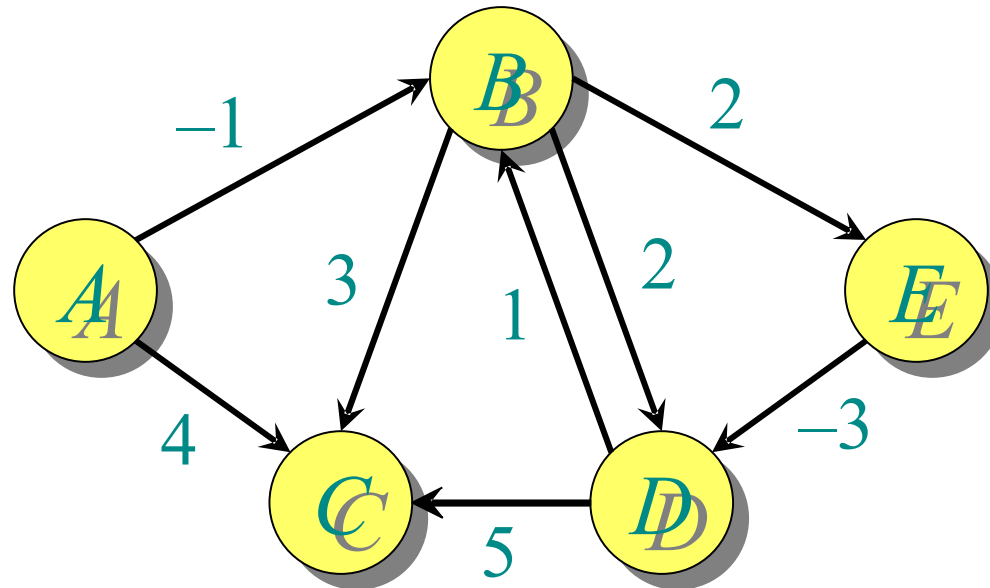
```
for  $i \leftarrow 1$  to  $|V| - 1$   
do for each edge  $(u, v) \in E$   
do if  $d[v] > d[u] + w(u, v)$   
then  $d[v] \leftarrow d[u] + w(u, v)$  } relaxation step
```

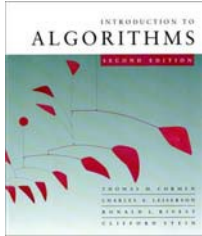
```
for each edge  $(u, v) \in E$   
do if  $d[v] > d[u] + w(u, v)$   
then report that a negative-weight cycle exists
```

At the end, $d[v] = \delta(s, v)$, if no negative-weight cycles.
Time = $O(VE)$.

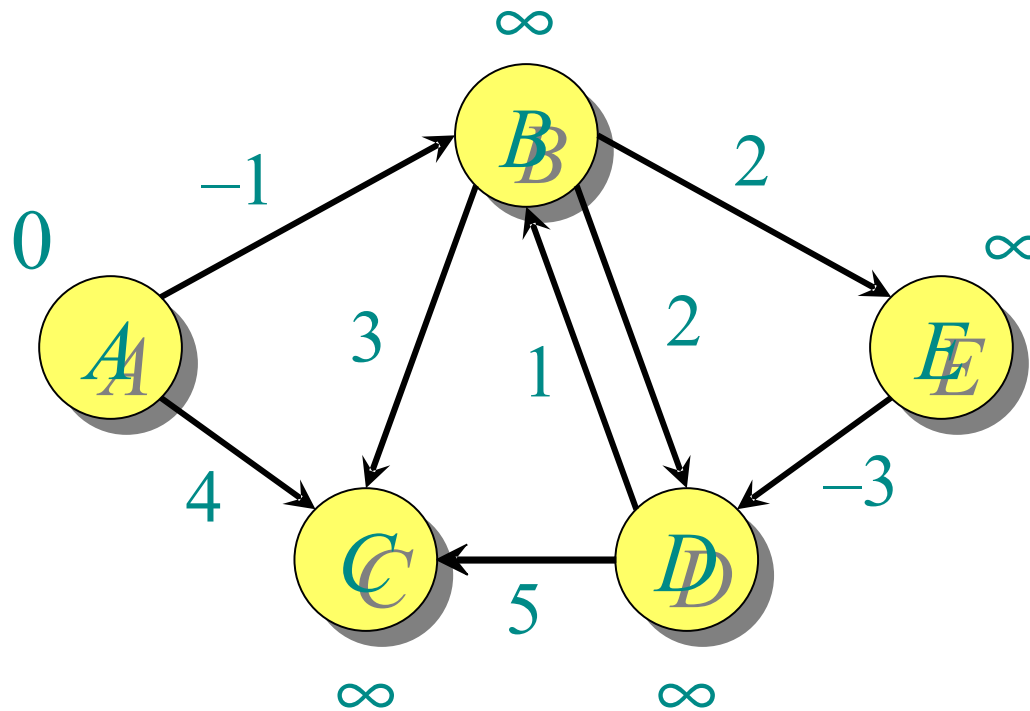


Example of Bellman-Ford

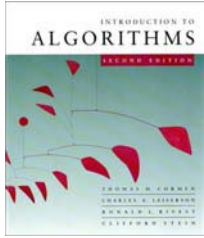




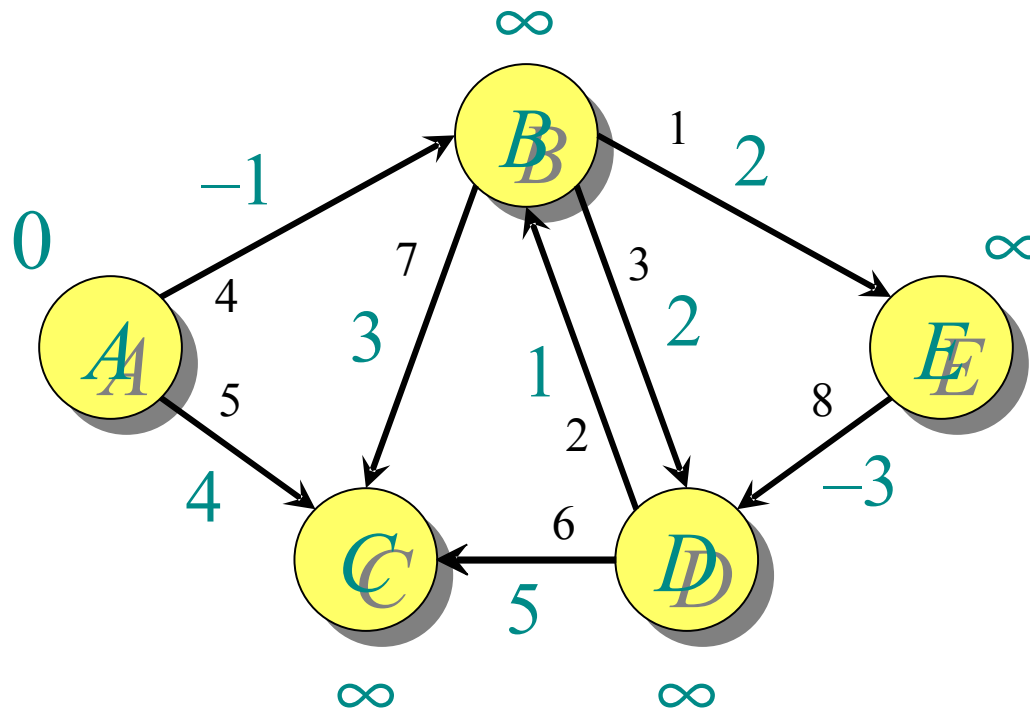
Example of Bellman-Ford



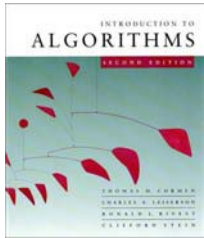
Initialization.



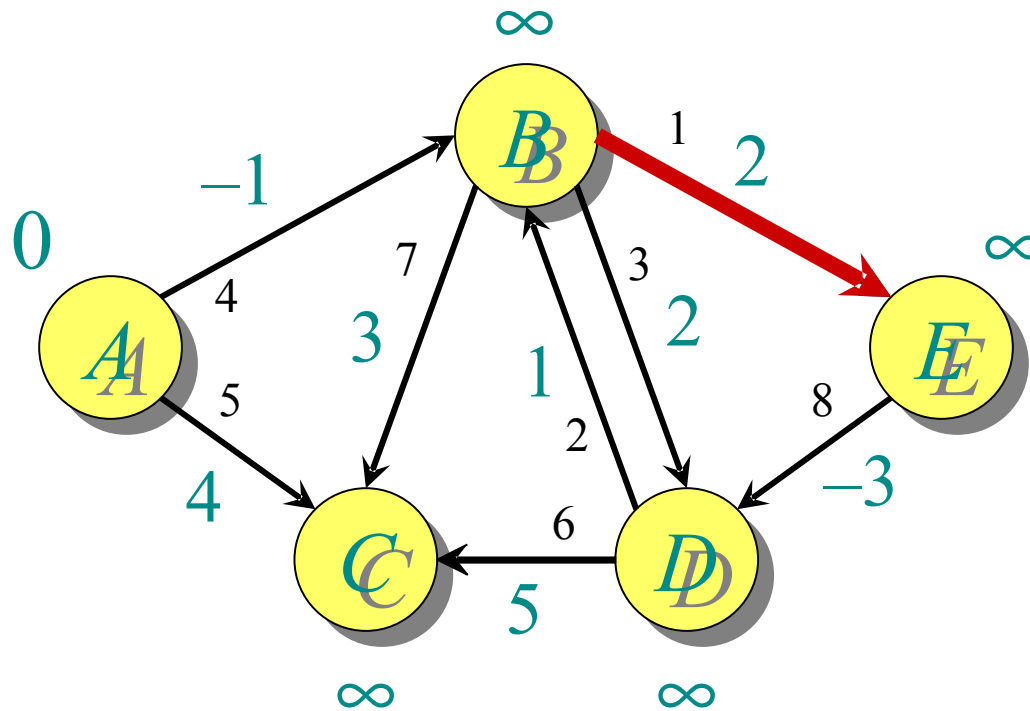
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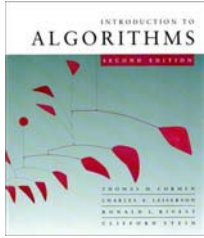


Order of edge relaxation.

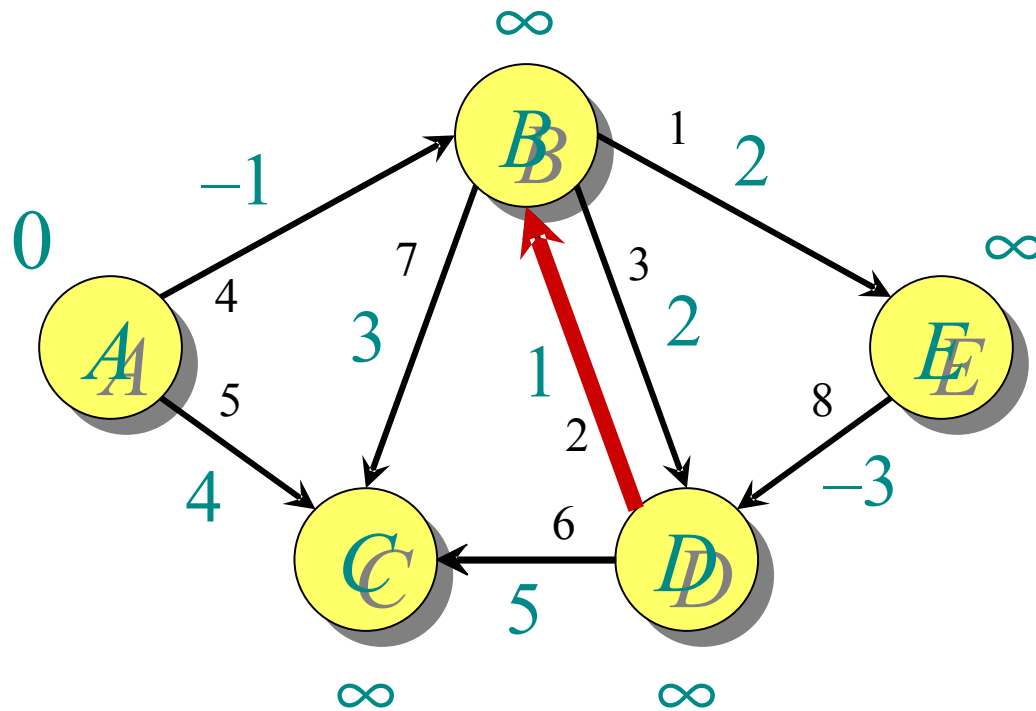


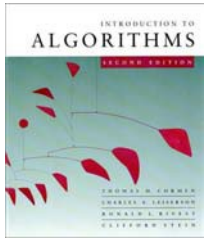
Example of Bellman-Ford



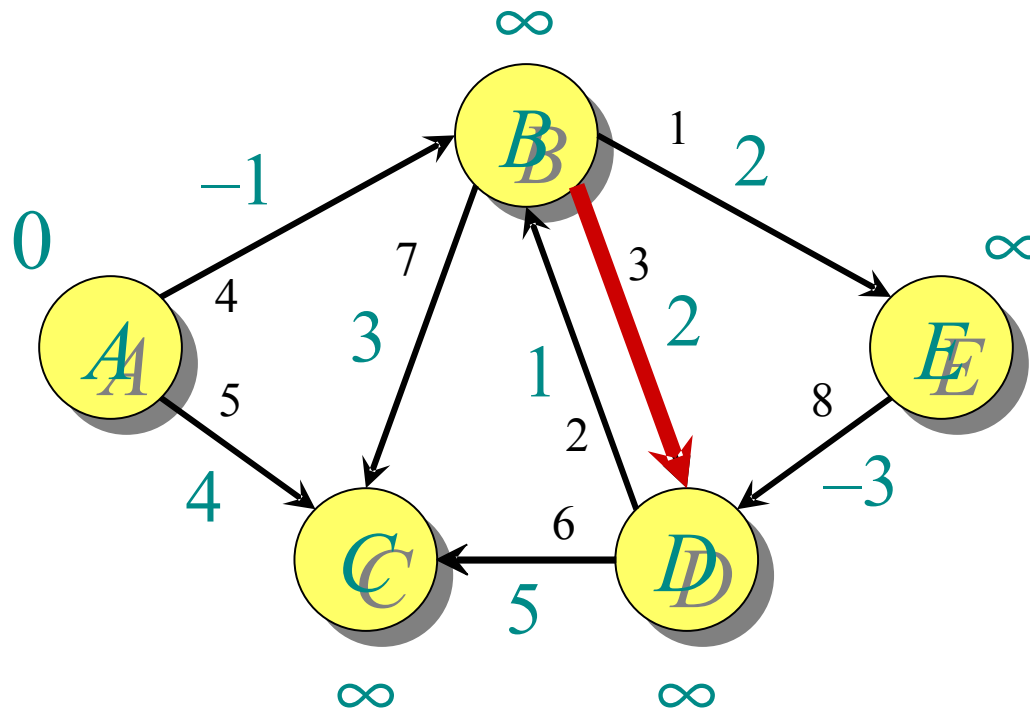


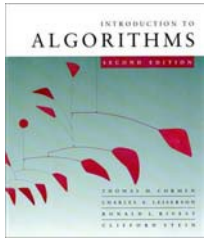
Example of Bellman-Ford



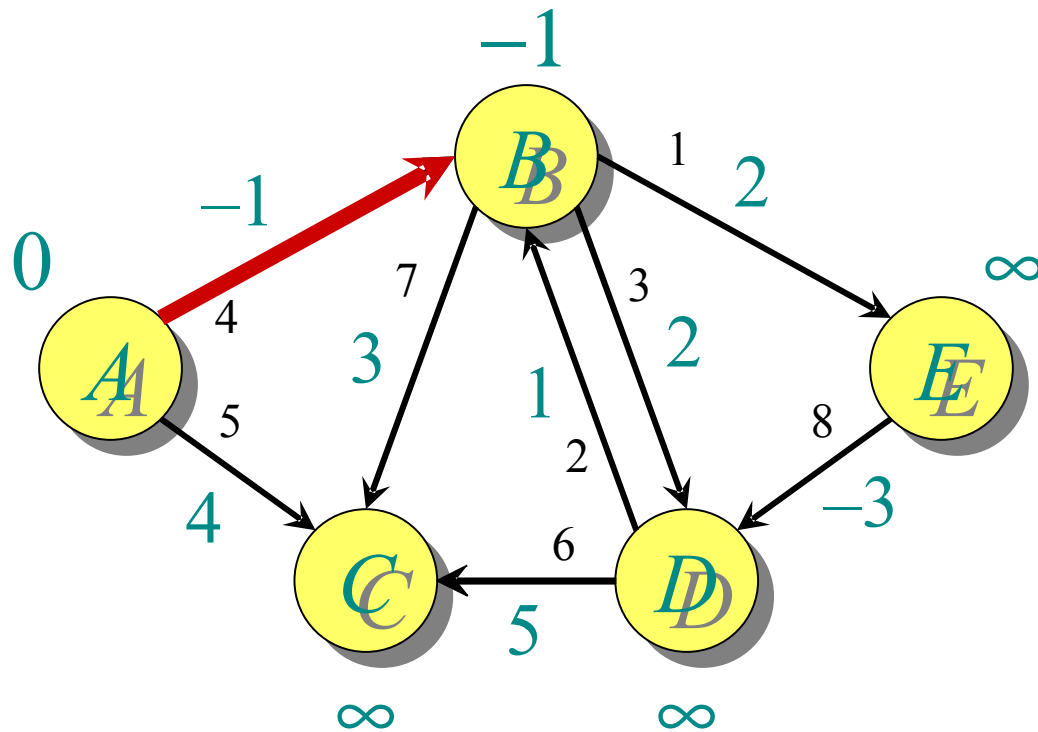


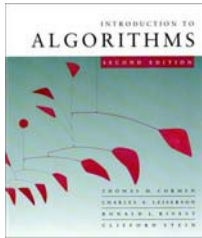
Example of Bellman-Ford



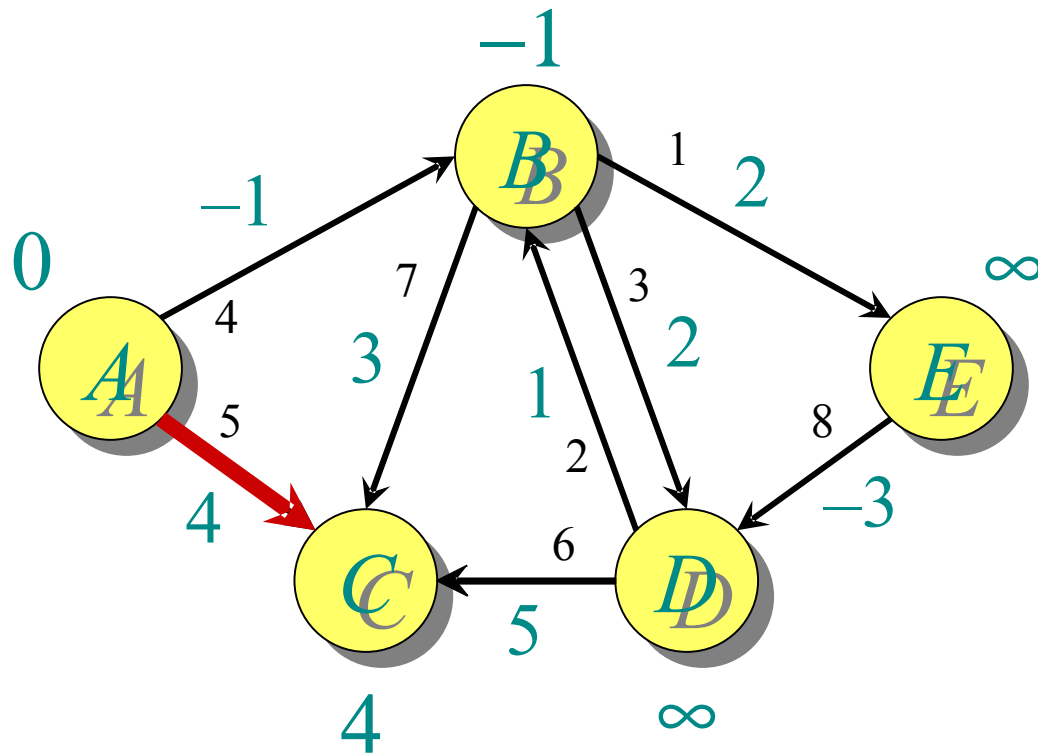


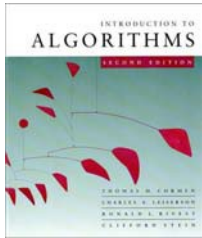
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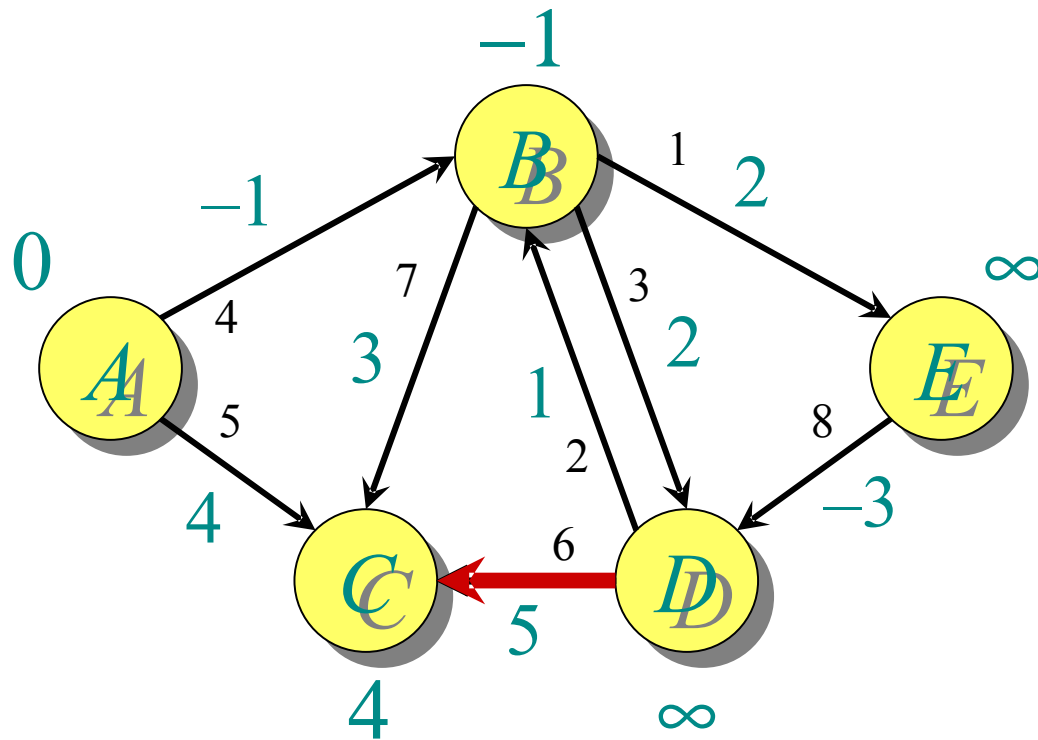


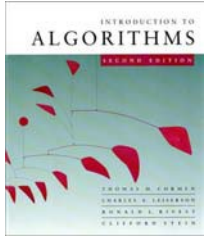
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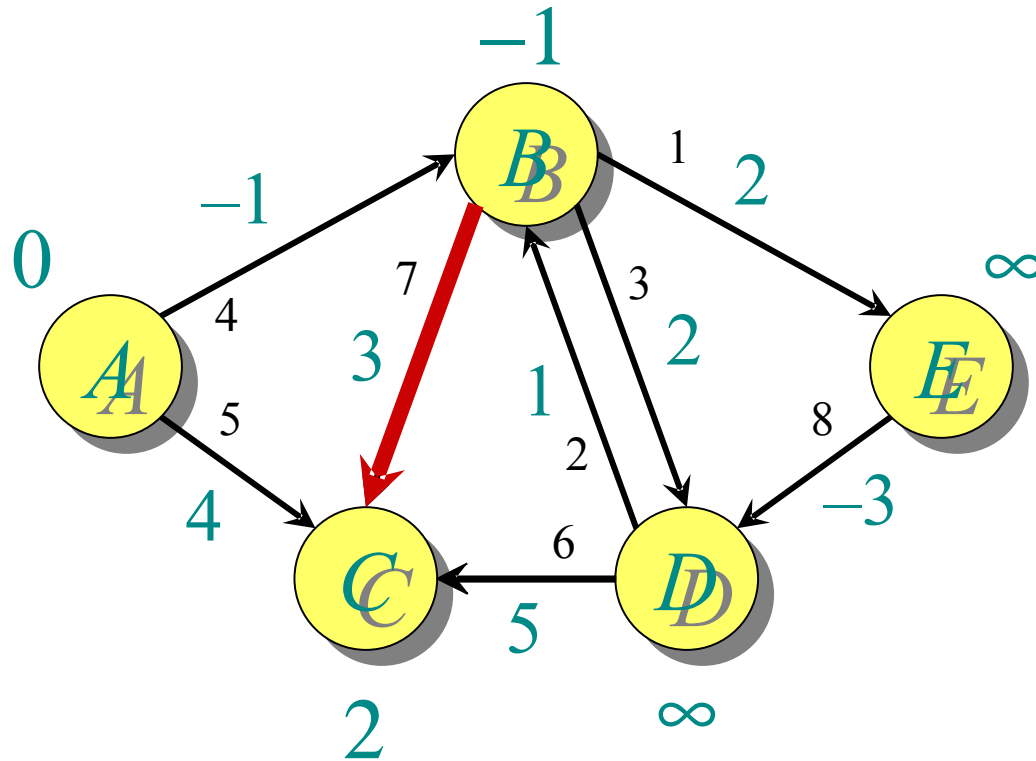


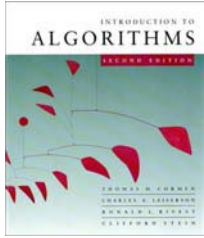
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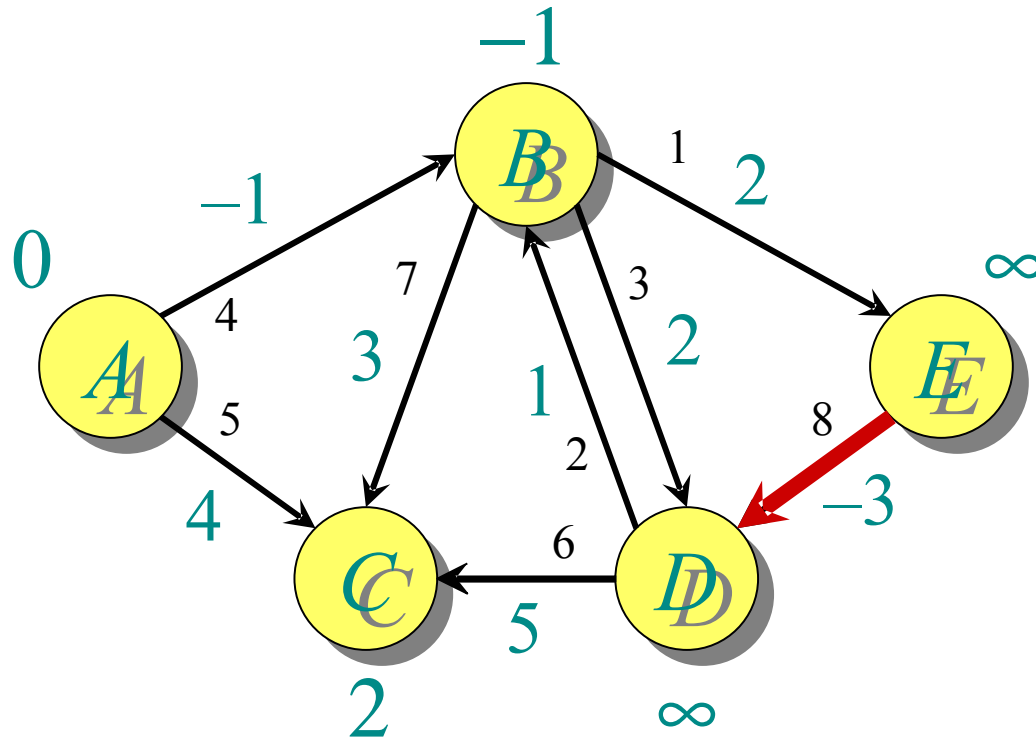


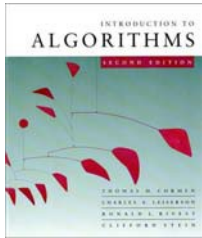
Example of Bellman-Ford



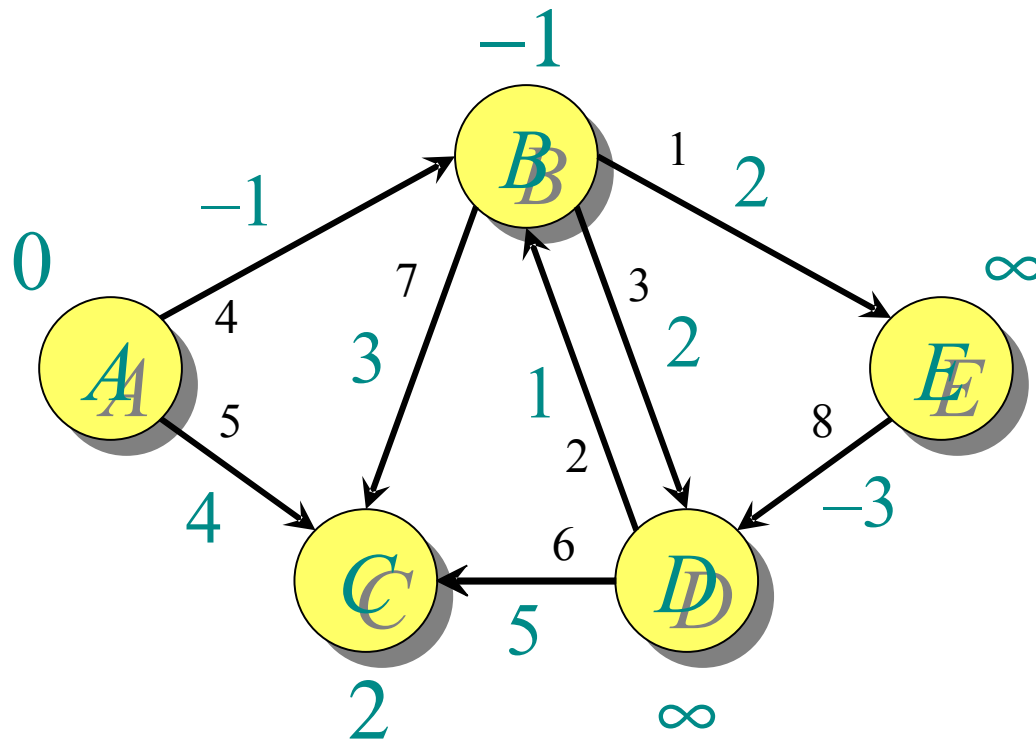


Example of Bellman-Ford

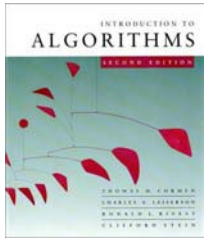




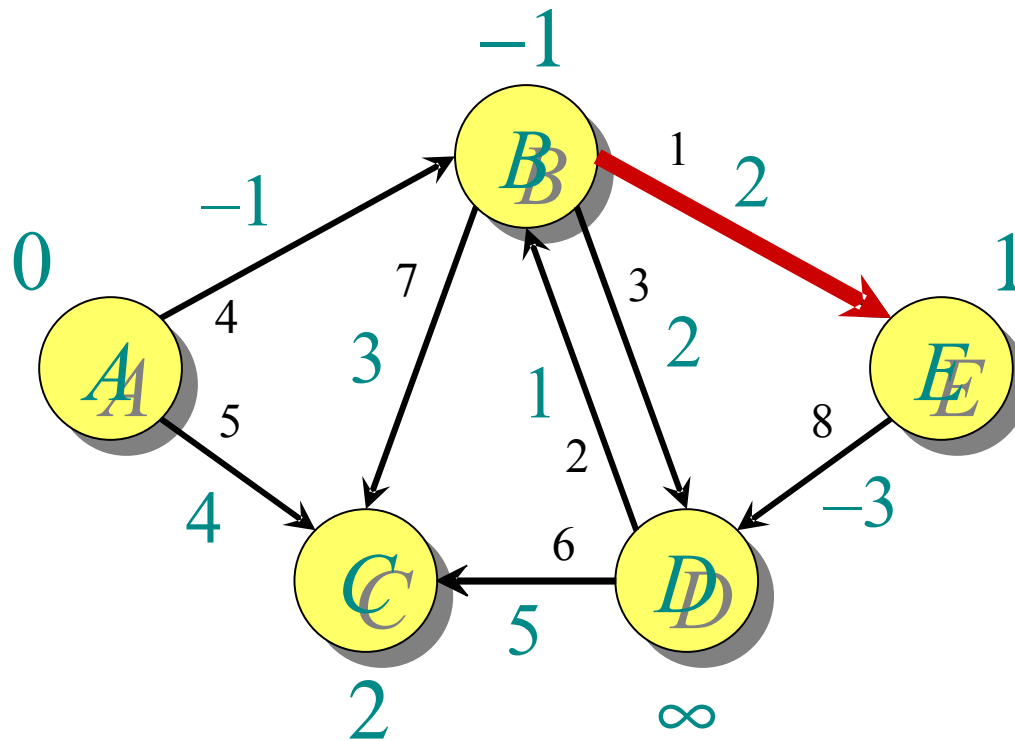
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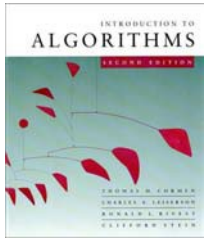


End of pass 1.

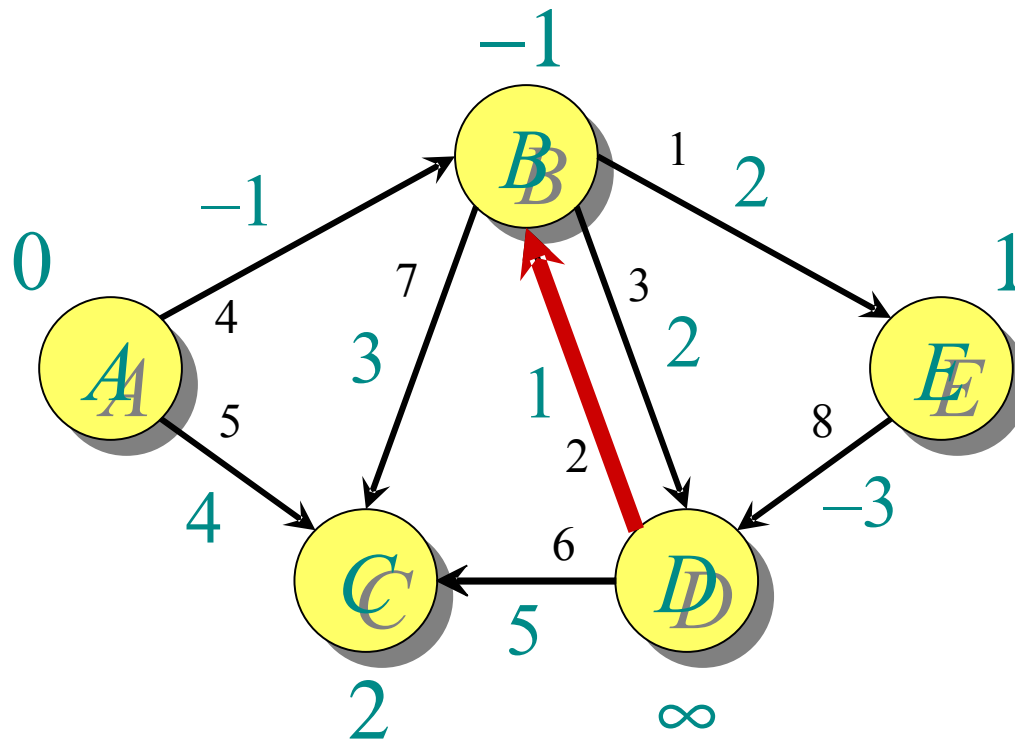


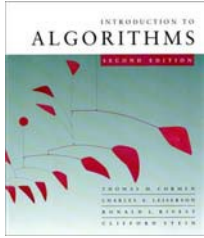
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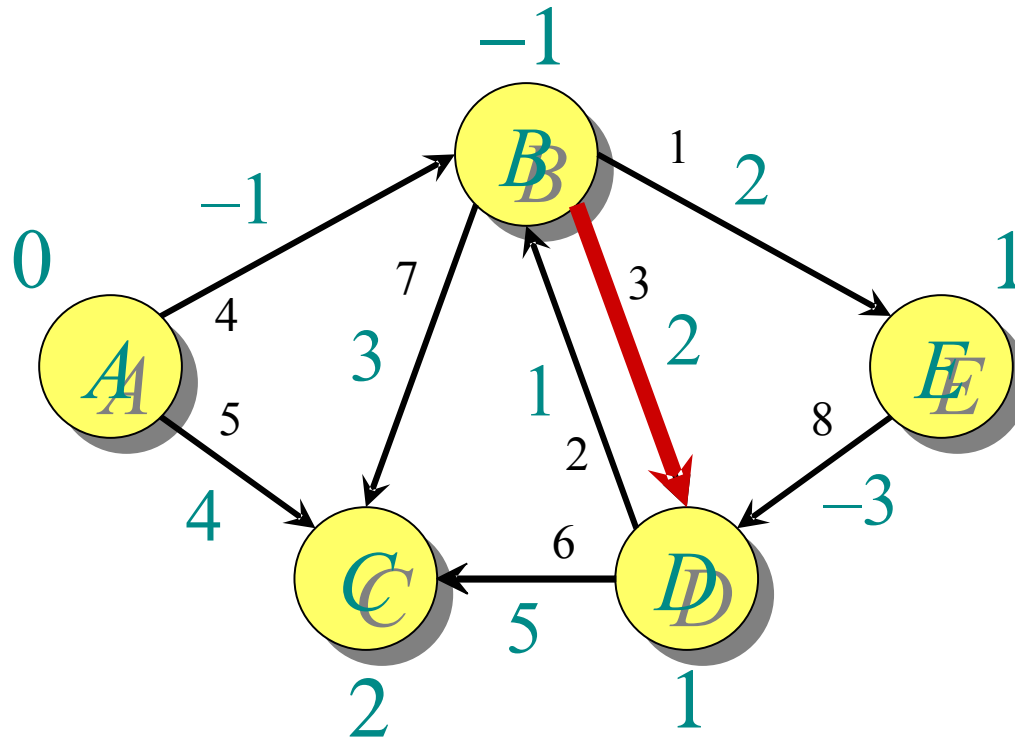


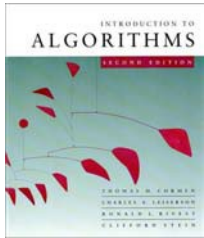
Example of Bellman-Ford



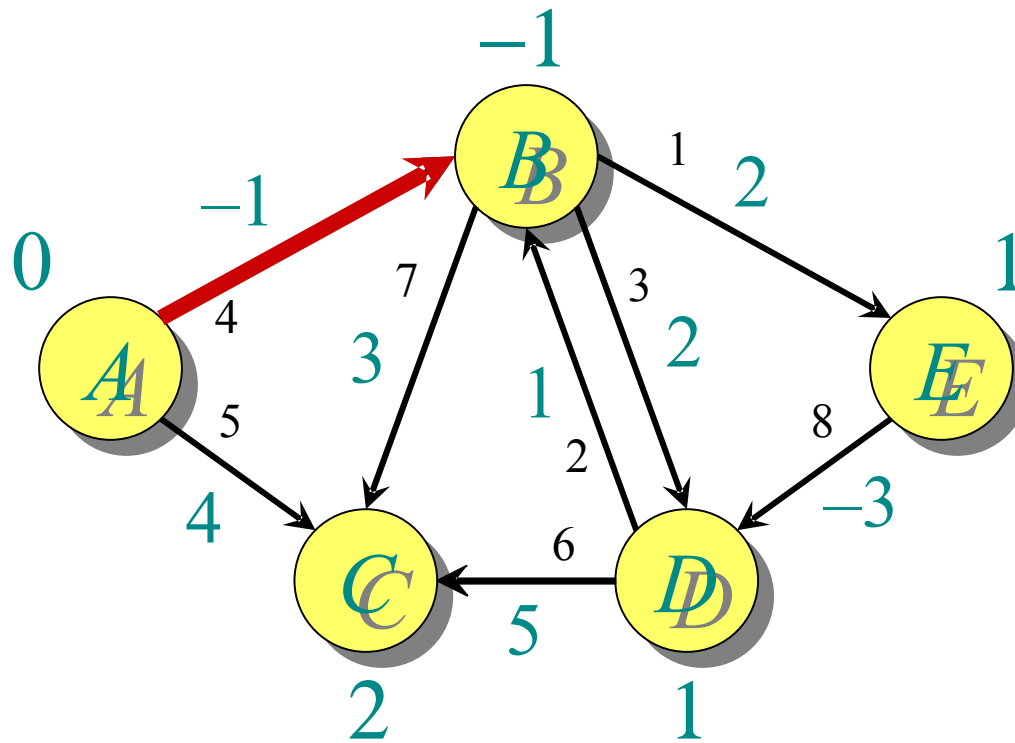


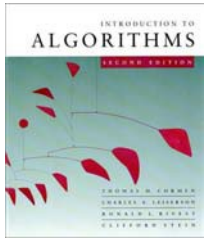
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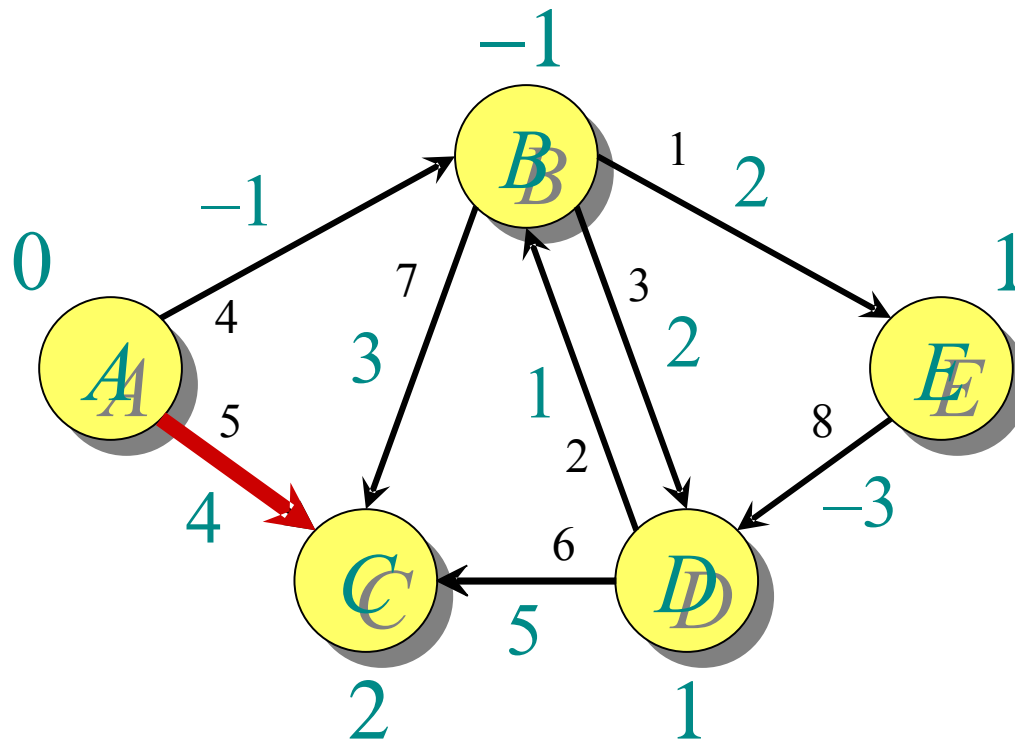


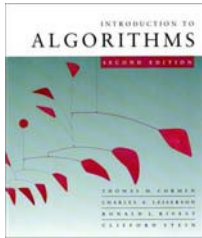
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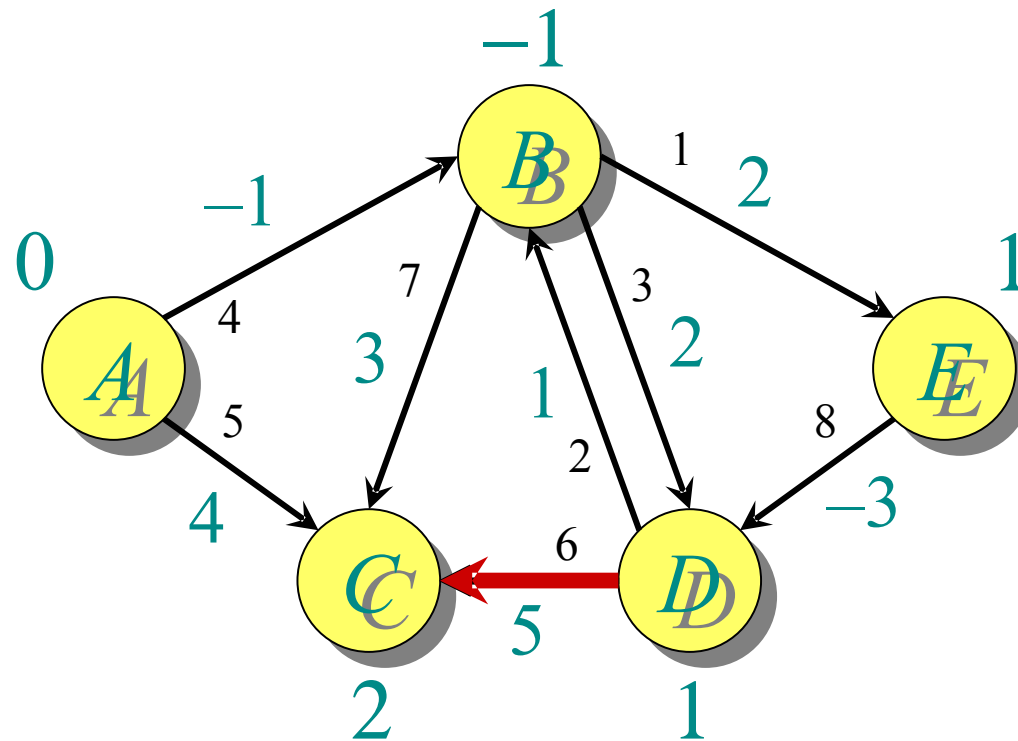


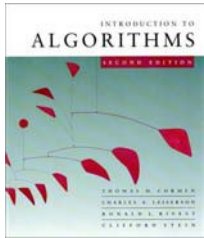
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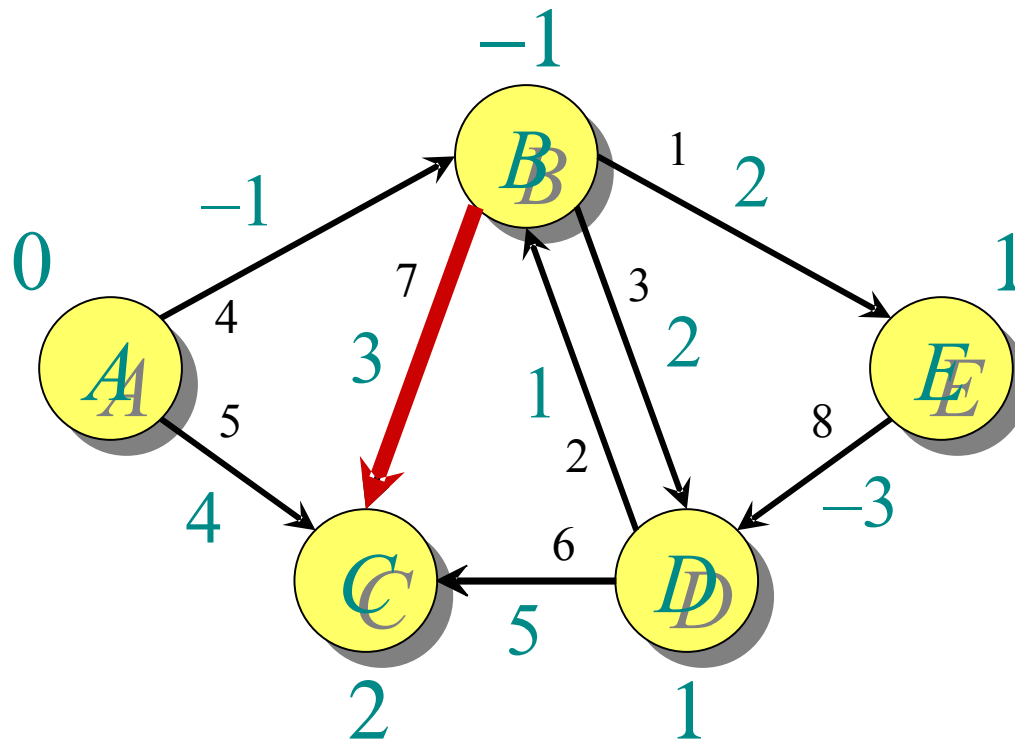


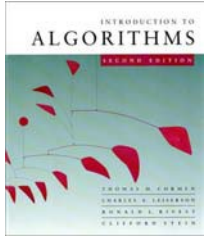
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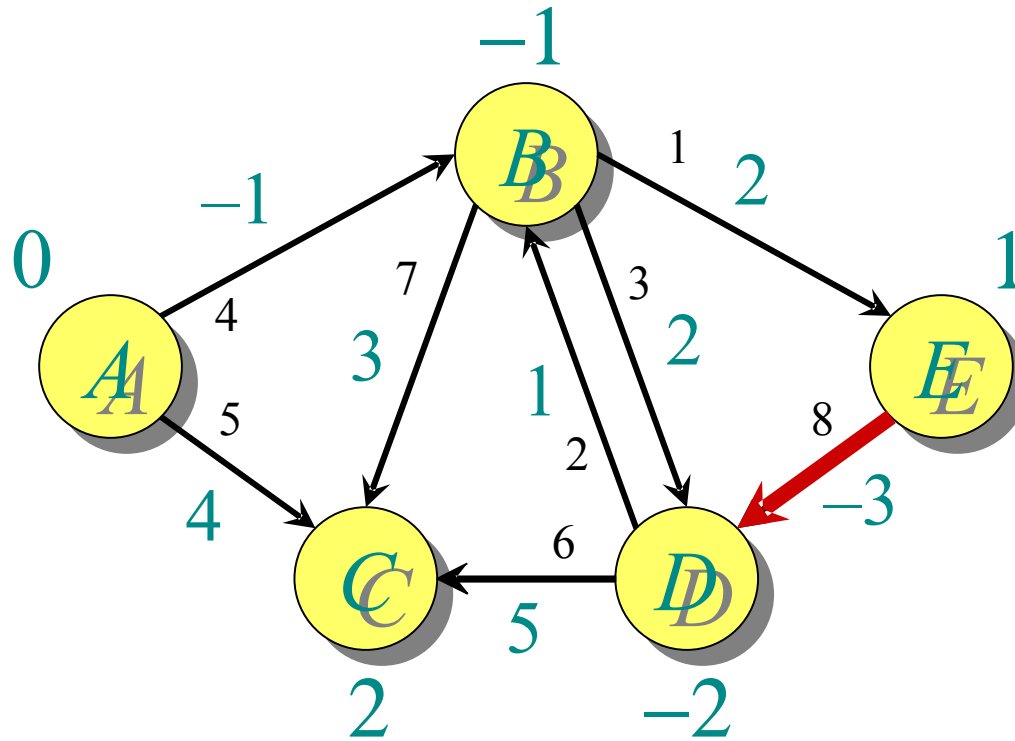


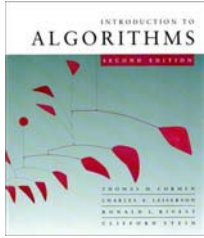
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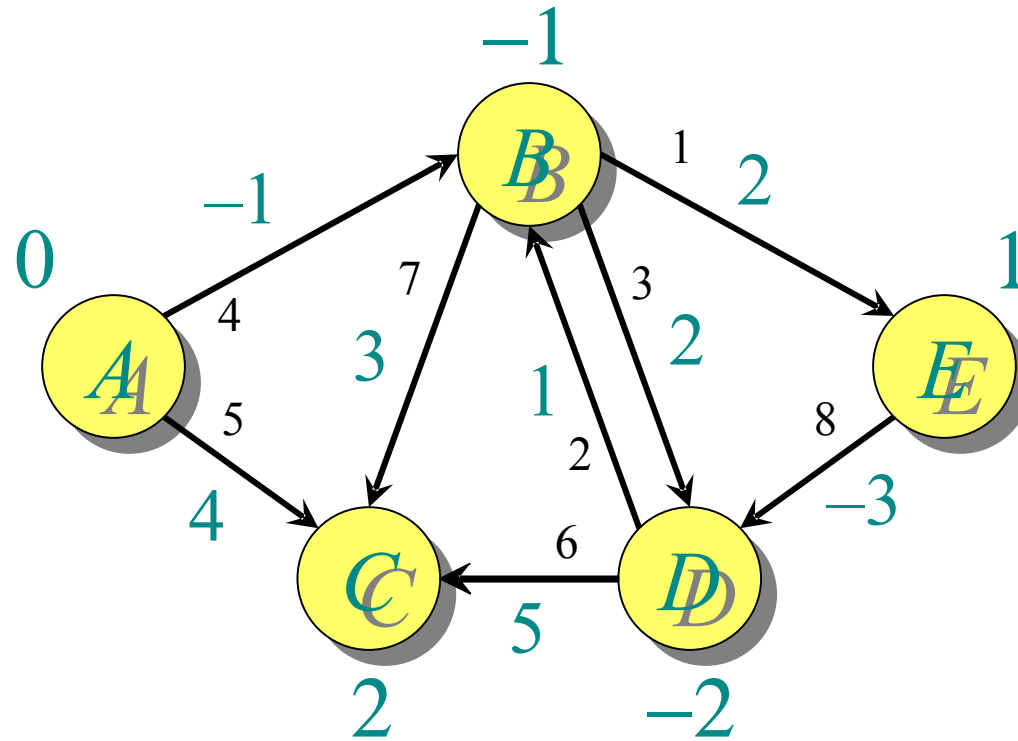


Example of Bellman-Ford

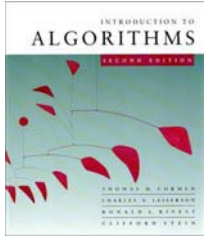




Example of Bellman-Ford

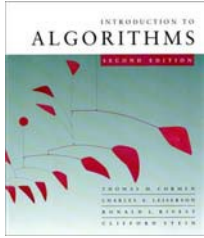


End of pass 2 (and 3 and 4).



Correctness

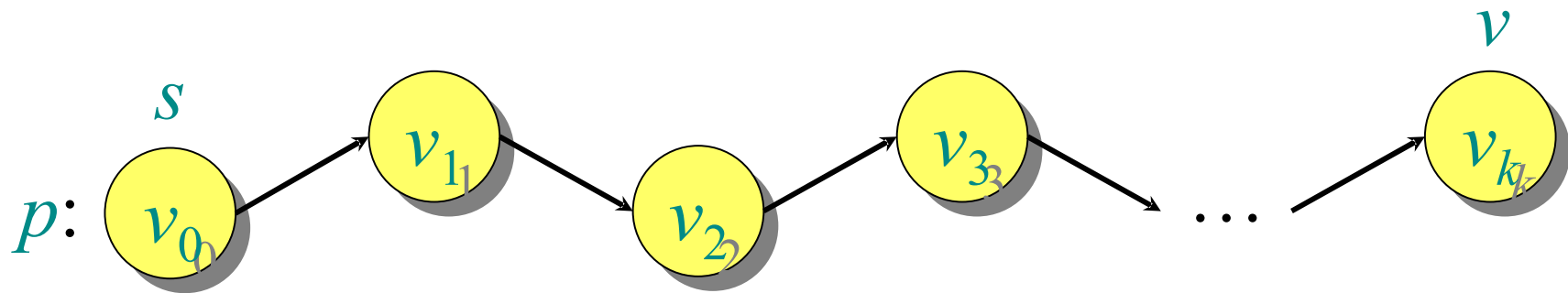
Theorem. If $G = (V, E)$ contains no negative-weight cycles, then after the Bellman-Ford algorithm executes, $d[v] = \delta(s, v)$ for all $v \in V$.



Correctness

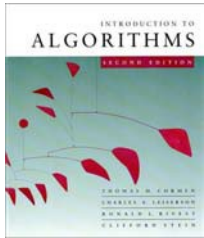
Theorem. If $G = (V, E)$ contains no negative-weight cycles, then after the Bellman-Ford algorithm executes, $d[v] = \delta(s, v)$ for all $v \in V$.

Proof. Let $v \in V$ be any vertex, and consider a shortest path p from s to v with the minimum number of edges.

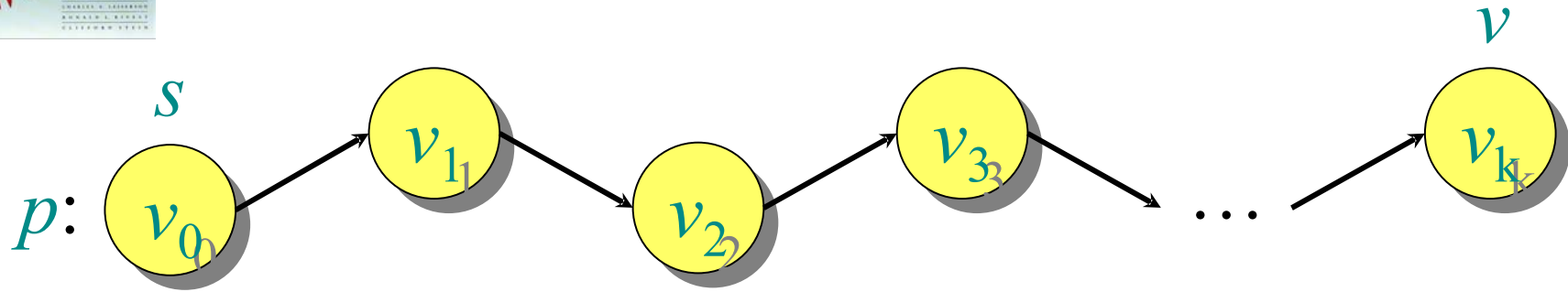


Since p is a shortest path, we have

$$\delta(s, v_i) = \delta(s, v_{i-1}) + w(v_{i-1}, v_i) .$$



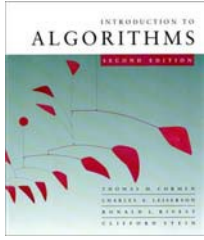
Correctness (continued)



Initially, $d[v_0] = 0 = \delta(s, v_0)$, and $d[v_0]$ is unchanged by subsequent relaxations (because of the lemma $d[v] \geq \delta(s, v)$)

- After 1 pass through E , we have $d[v_1] = \delta(s, v_1)$.
- After 2 passes through E , we have $d[v_2] = \delta(s, v_2)$.
- \vdots
- After k passes through E , we have $d[v_k] = \delta(s, v_k)$.

Since G contains no negative-weight cycles, p is simple. Longest simple path has $\leq |V| - 1$ edges. \square



Detection of negative-weight cycles

Corollary. If a value $d[v]$ fails to converge after $|V| - 1$ passes, there exists a negative-weight cycle in G reachable from s . □

DAG Shortest Paths

Bellman-Ford takes time $O(VE)$. Can we do better on a DAG?

Yes!

How?

We can apply *topological sorting* on the DAG, and *relax* every edge only *once*.

DAG Shortest Paths

$\Theta(E+V)$ { Topological-Sort(G)

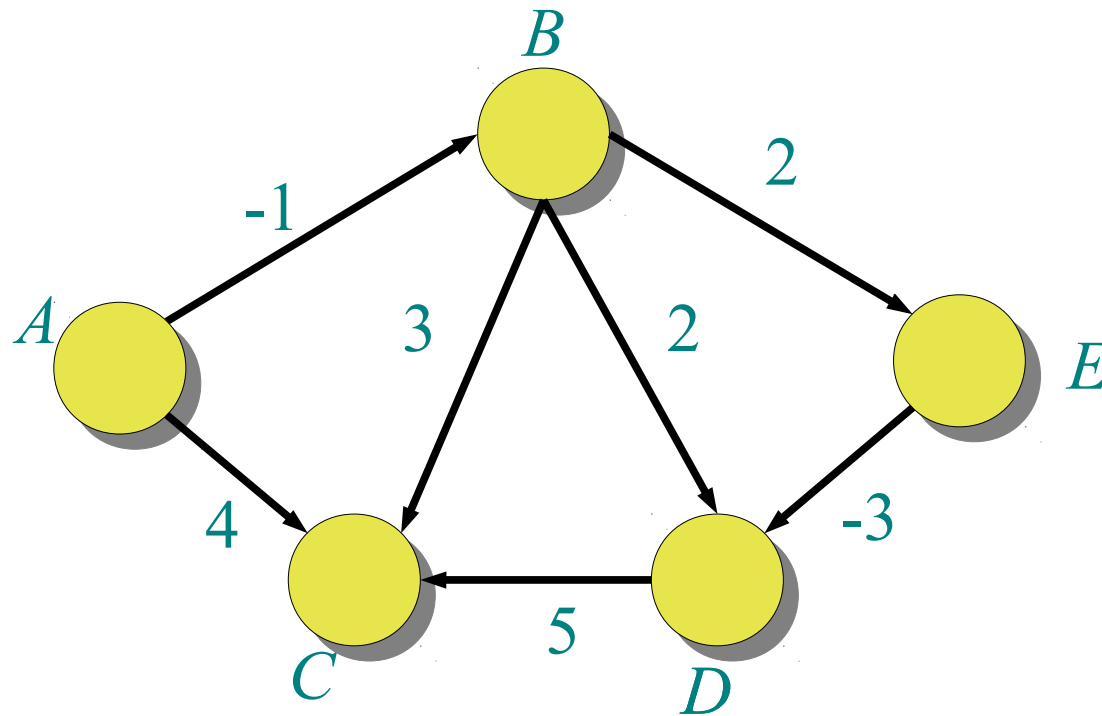
$\Theta(V)$ { for v in V
 $d[v] = \infty$
 $d[s] = 0$

$\Theta(E)$ { for each vertex u in topologically sorted order
for each vertex v in $Adj[u]$
if $d[v] > d[u] + w(u,v)$
 $d[v] = d[u] + w(u,v)$

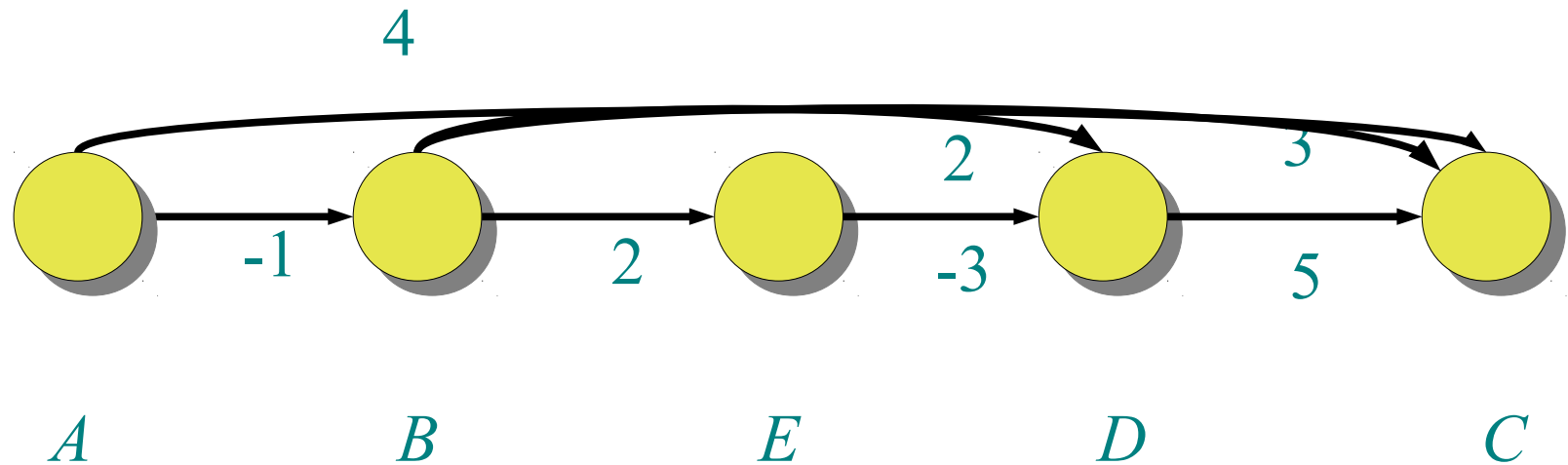
Total Time = $\Theta(E+V)$

Proof. Exercise.

DAG Shortest Paths

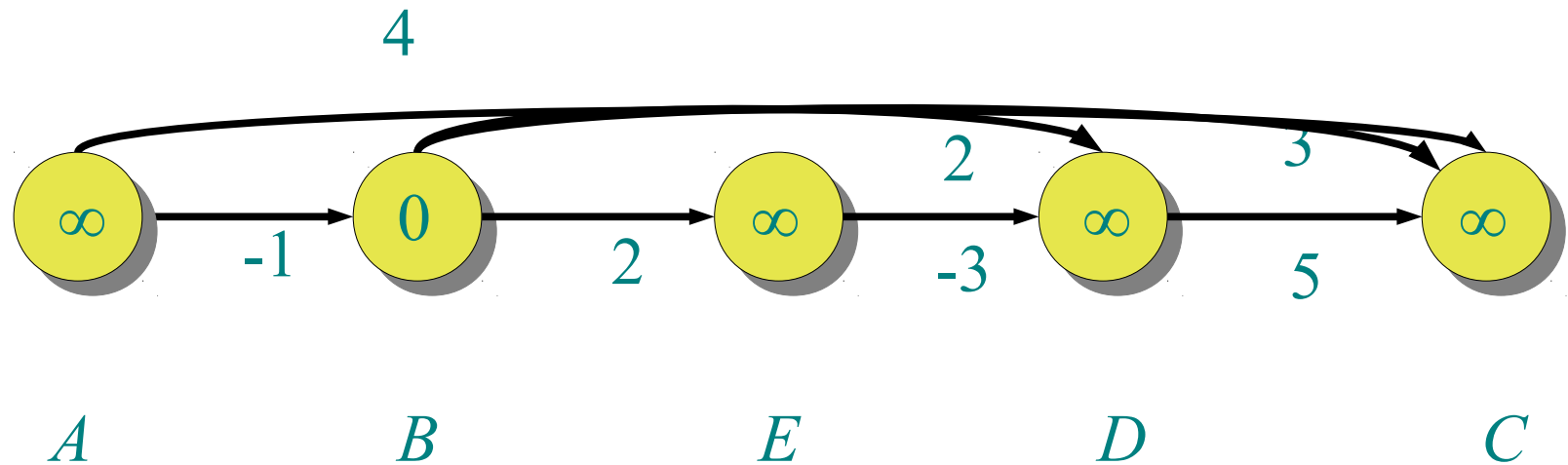


DAG Shortest Paths



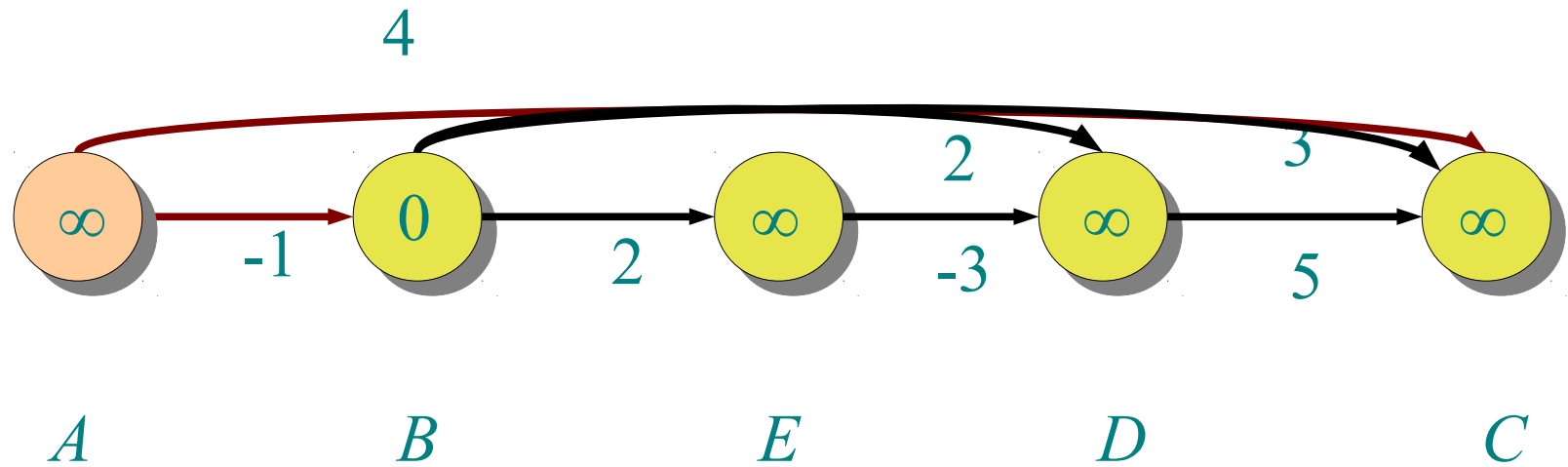
Topological Sorting

DAG Shortest Paths



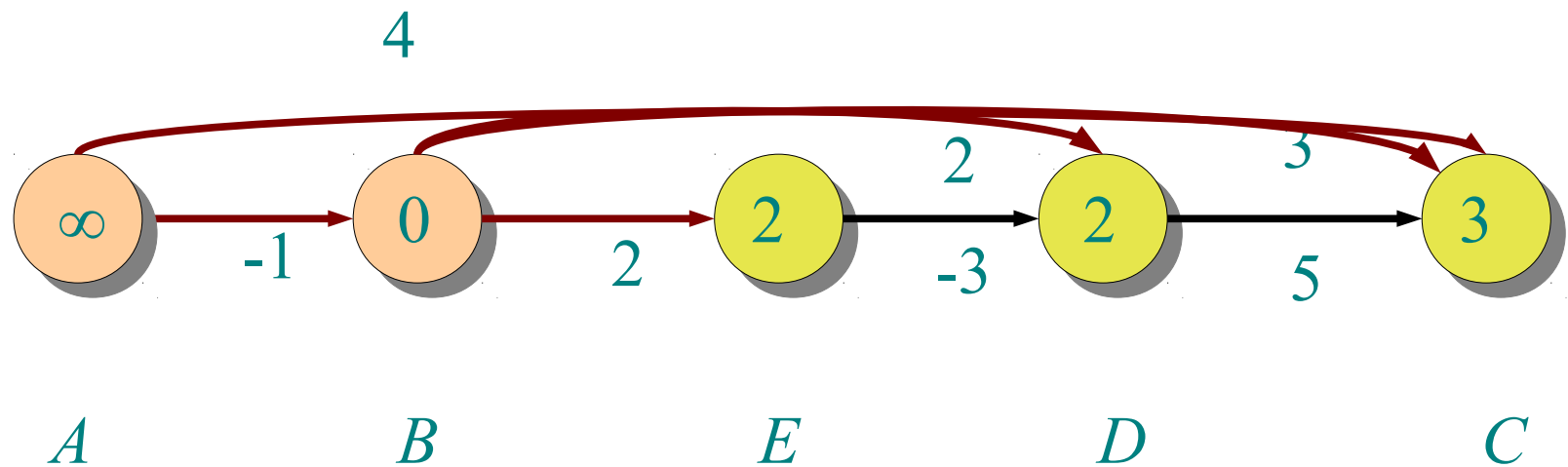
Initialize with *B* as a source

DAG Shortest Paths



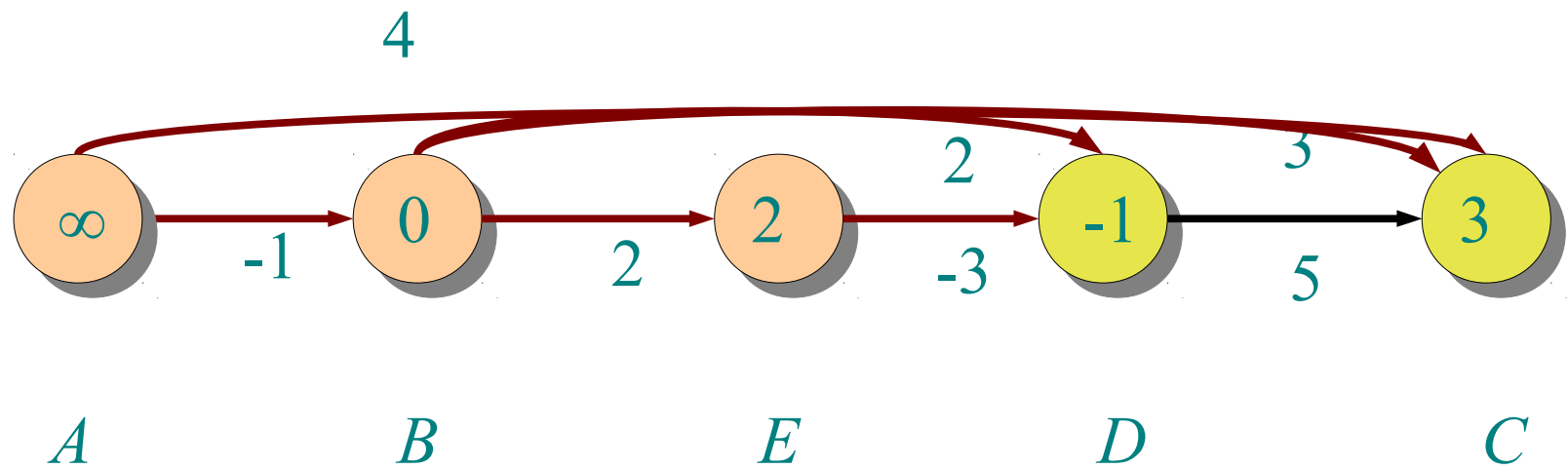
Process *A*

DAG Shortest Paths



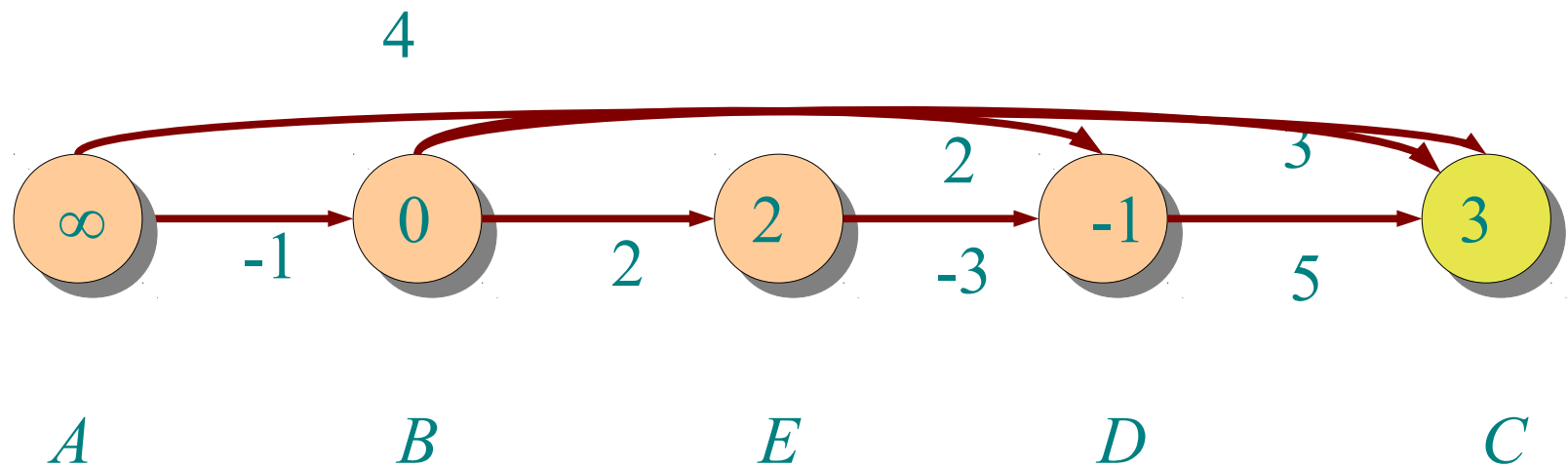
Process *B*

DAG Shortest Paths



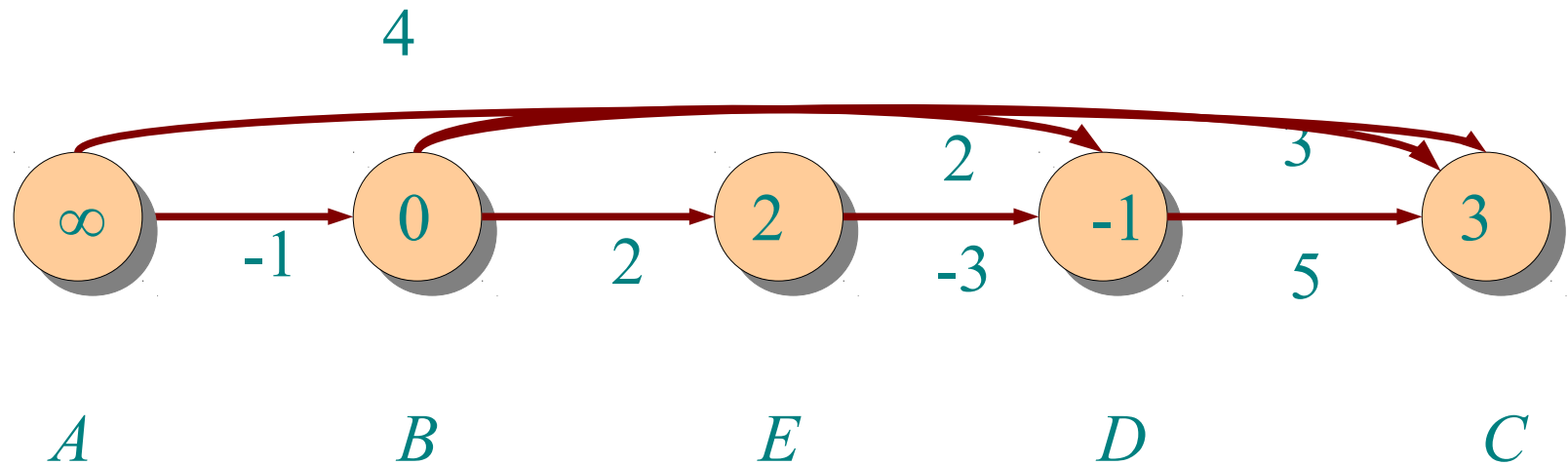
Process *E*

DAG Shortest Paths

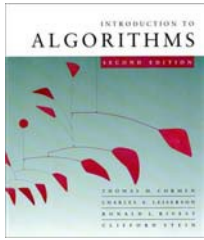


Process *D*

DAG Shortest Paths

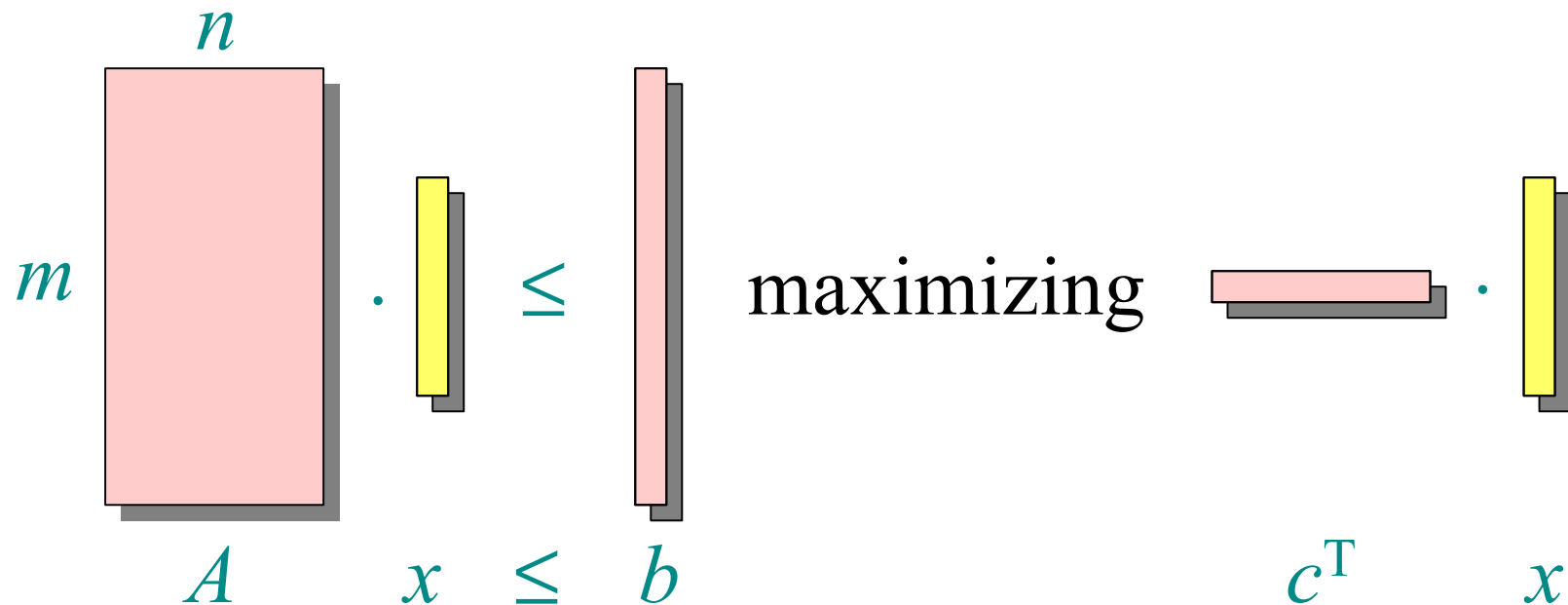


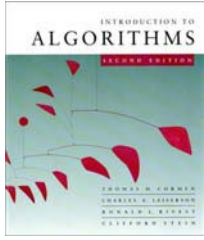
Process *C*



Linear programming

Let A be an $m \times n$ matrix, b be an m -vector, and c be an n -vector. Find an n -vector x that maximizes $c^T x$ subject to $Ax \leq b$, or determine that no such solution exists.

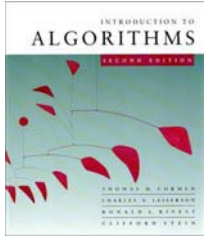




Linear-programming algorithms

Algorithms for the general problem

- Simplex methods — practical, but worst-case exponential time.
- Interior-point methods — polynomial time and competes with simplex.



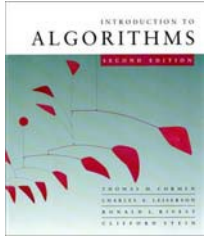
Linear-programming algorithms

Algorithms for the general problem

- Simplex methods — practical, but worst-case exponential time.
- Interior-point methods — polynomial time and competes with simplex.

Feasibility problem: No optimization criterion. Just find x such that $Ax \leq b$.

- In general, just as hard as ordinary LP.

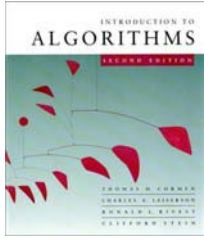


Solving a system of difference constraints

Linear programming where each row of A contains exactly one 1 , one -1 , and the rest 0 's.

Example:

$$\left. \begin{array}{l} x_1 - x_2 \leq 3 \\ x_2 - x_3 \leq -2 \\ x_1 - x_3 \leq 2 \end{array} \right\} x_j - x_i \leq w_{ij}$$



Solving a system of difference constraints

Linear programming where each row of A contains exactly one 1 , one -1 , and the rest 0 's.

Example:

$$x_1 - x_2 \leq 3$$

$$x_2 - x_3 \leq -2$$

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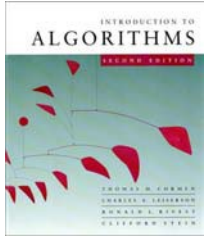
$$\left. \begin{array}{l} x_1 - x_2 \leq 3 \\ x_2 - x_3 \leq -2 \\ x_1 - x_3 \leq 2 \end{array} \right\} x_j - x_i \leq w_{ij}$$

Solution:

$$x_1 = 3$$

$$x_2 = 0$$

$$x_3 = 2$$



Solving a system of difference constraints

Linear programming where each row of A contains exactly one 1, one -1 , and the rest 0's.

Example:

$$x_1 - x_2 \leq 3$$

$$x_2 - x_3 \leq -2$$

$$x_1 - x_3 \leq 2$$

$$x_j - x_i \leq w_{ij}$$

Solution:

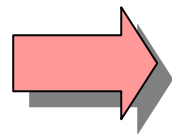
$$x_1 = 3$$

$$x_2 = 0$$

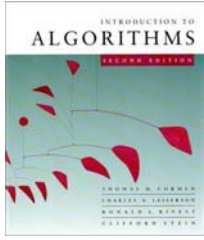
$$x_3 = 2$$

Constraint graph:

$$x_j - x_i \leq w_{ij}$$

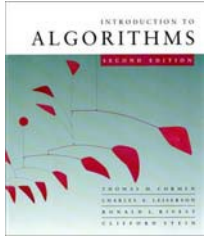


(The “ A ” matrix has dimensions $|E| \times |V|$.)



Unsatisfiable constraints

Theorem. If the constraint graph contains a negative-weight cycle, then the system of differences is unsatisfiable.

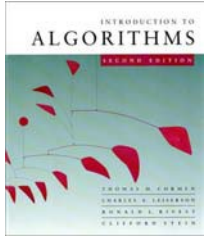


Unsatisfiable constraints

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Proof. Suppose that the negative-weight cycle is $v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_k \rightarrow v_1$. Then, we have

$$\begin{aligned}x_2 - x_1 &\leq w_{12} \\x_3 - x_2 &\leq w_{23} \\&\vdots \\x_k - x_{k-1} &\leq w_{k-1, k} \\x_1 - x_k &\leq w_{k1}\end{aligned}$$



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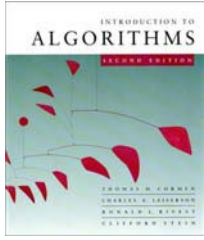
$$\vdots$$

$$x_k - x_{k-1} \leq w_{k-1, k}$$

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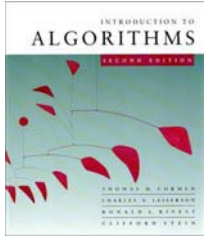
$$0 \leq \text{weight of cycle} < 0$$

Therefore, no values for the x_i can satisfy the constraints. □



Satisfying the constraints

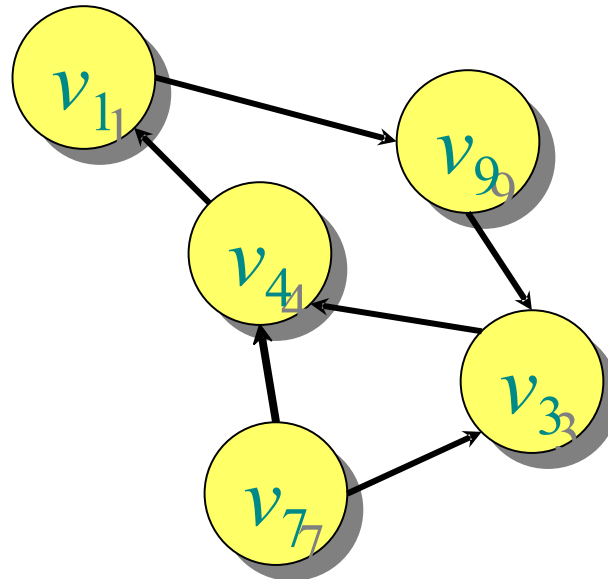
Theorem. Suppose no negative-weight cycle exists in the constraint graph. Then, the constraints are satisfiable.

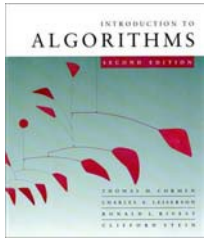


Satisfying the constraints

Theorem. Suppose no negative-weight cycle exists in the constraint graph. Then, the constraints are satisfiable.

Proof. Add a new vertex s to V with a 0-weight edge to each vertex $v_i \in V$.

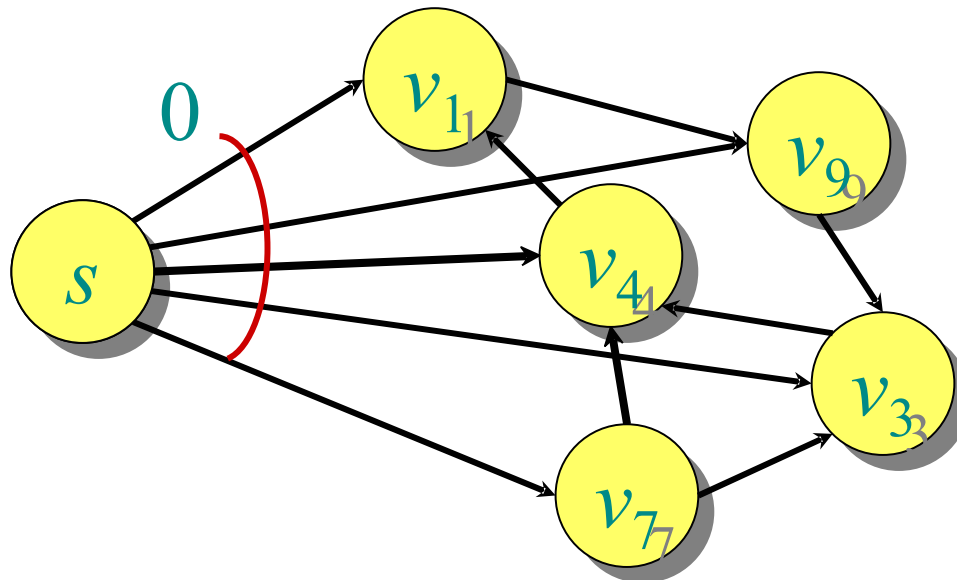




Satisfying the constraints

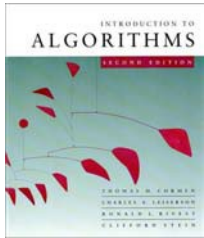
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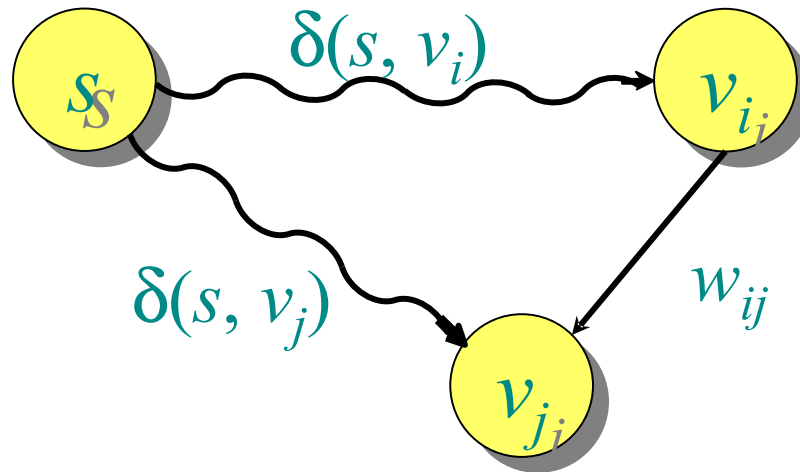
Note:

No negative-weight cycles introduced \Rightarrow shortest paths exist.

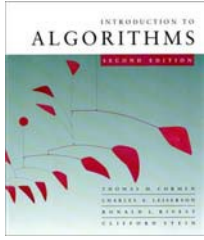


Proof (continued)

Claim: The assignment $x_i = \delta(s, v_i)$ solves the constraints. Consider any constraint $x_j - x_i \leq w_{ij}$, and consider the shortest paths from s to v_j and v_i :

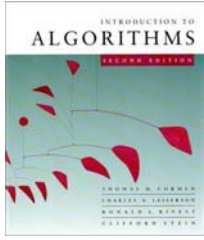


The triangle inequality gives us $\delta(s, v_j) \leq \delta(s, v_i) + w_{ij}$. Since $x_i = \delta(s, v_i)$ and $x_j = \delta(s, v_j)$, the constraint $x_j - x_i \leq w_{ij}$ is satisfied. □



Bellman-Ford and linear programming

Corollary. The Bellman-Ford algorithm can solve a system of m difference constraints on n variables in $O(n^2 + nm)$ time \square



Bellman-Ford and linear programming

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Proof.

The augmented constraint graph has $n+1$ vertices (variables) and $m+n$ edges. **why?**

Therefore, Bellman-Ford takes time

$$O(VE) = O((n+1)(m+n)) = O(n^2 + nm)$$

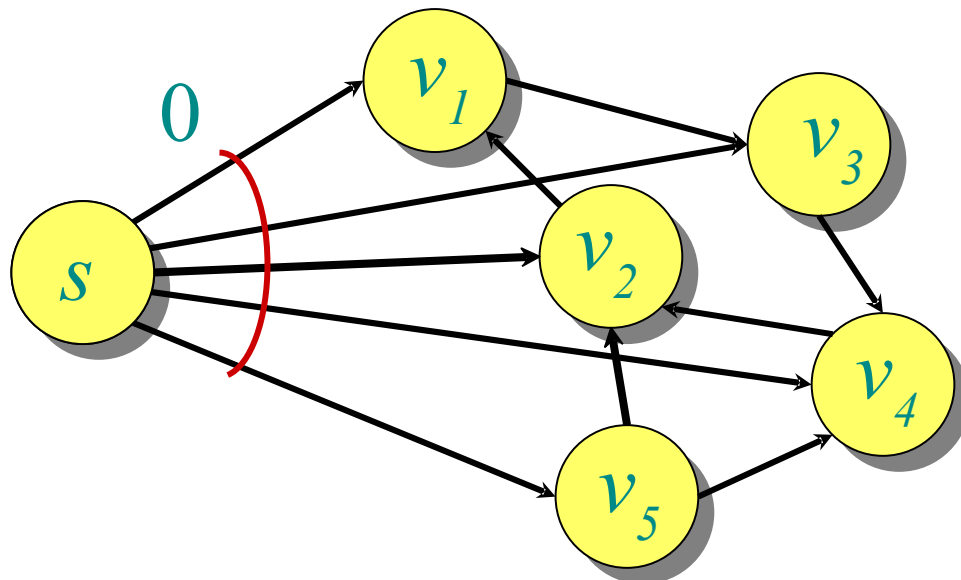
Can we make it take time $O(mn)$ instead? Yes!

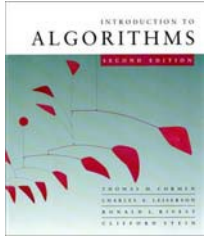
Bellman-Ford and Linear Programming

Notice that the *first* time any edge (s, v_i) is updated, $d[v_i]$ is set to 0 instead of ∞ , and it does *not* increase afterwards. *Why?*

So we can do away with s altogether, and initialize $d[v_i] = 0$

Therefore, we will have n vertices and m edges, and Bellman-Ford will take time $O(mn)$





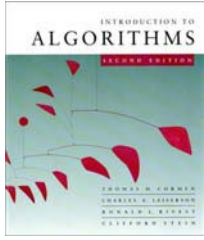
Bellman-Ford and linear programming

Corollary. The Bellman-Ford algorithm can solve a system of m difference constraints on n variables in $O(mn)$ time. \square

Single-source shortest paths is a simple LP problem.

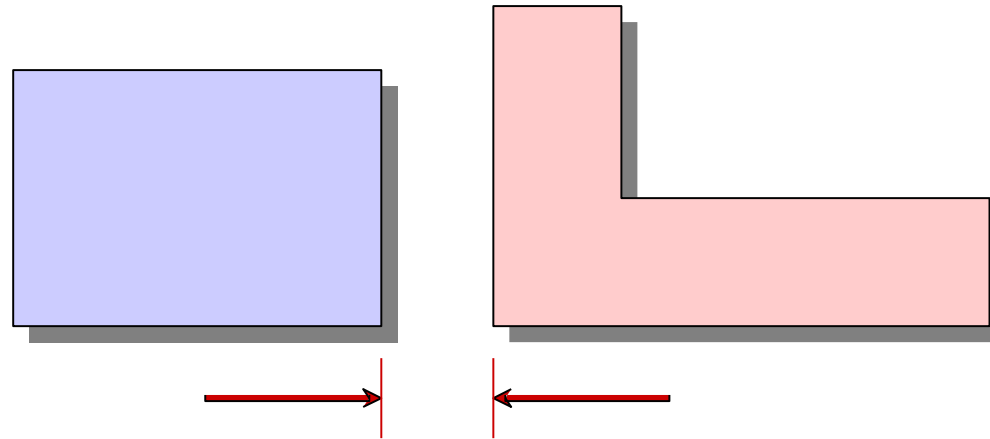
In fact, Bellman-Ford maximizes $x_1 + x_2 + \dots + x_n$ subject to the constraints $x_j - x_i \leq w_{ij}$ and $x_i \leq 0$ (exercise).

Bellman-Ford also minimizes $\max_i \{x_i\} - \min_i \{x_i\}$ (exercise).



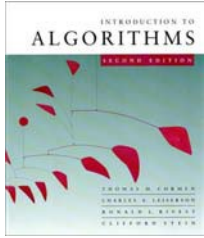
Application to VLSI layout compaction

*Integrated
-circuit
features:*

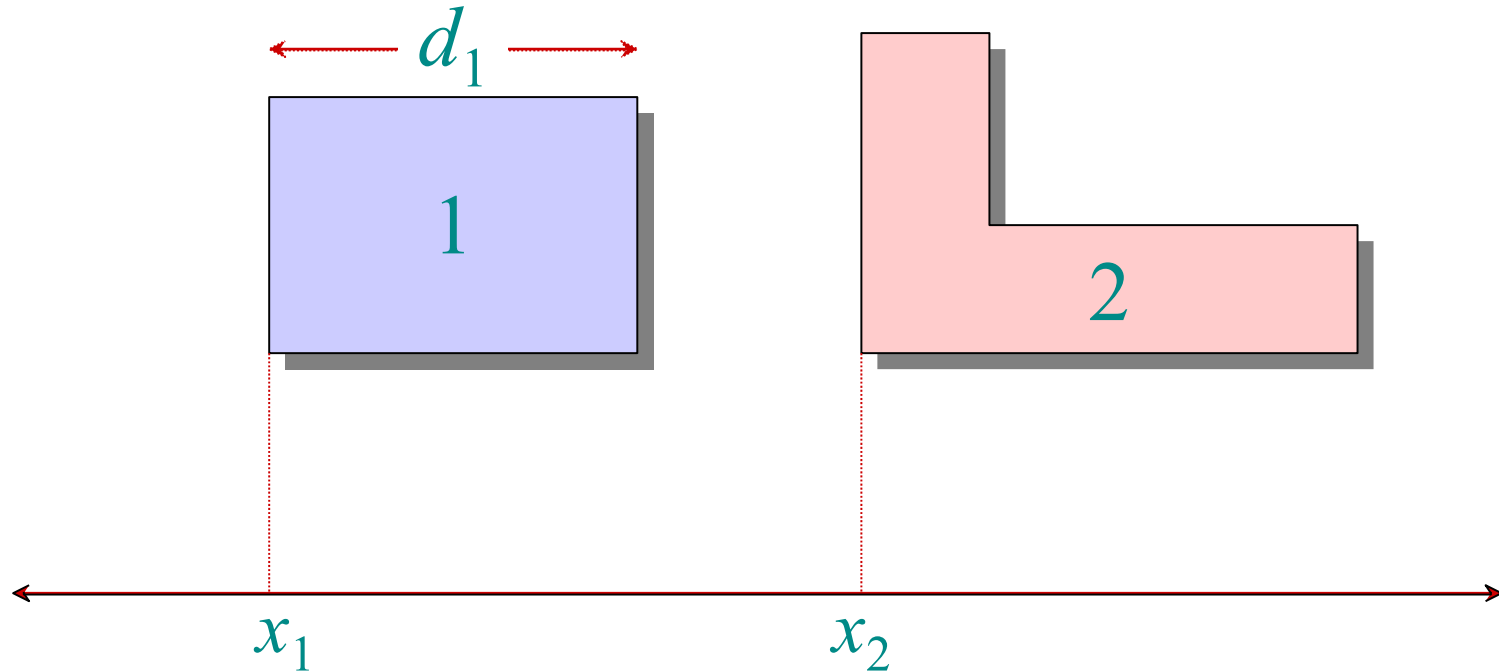


minimum separation λ

Problem: Compact (in one dimension) the space between the features of a VLSI layout without bringing any features too close together.



VLSI layout compaction



Constraint: $x_2 - x_1 \geq d_1 + \lambda$

Bellman-Ford minimizes $\max_i \{x_i\} - \min_i \{x_i\}$, which compacts the layout in the x -dimension.

Recap

- Bellman-Ford Algorithm
- DAG Shortest Paths
- Linear Programming and Difference Constraints
- VLSI Layout Compaction