CMP302: Algorithms

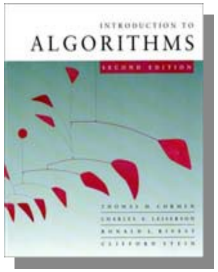
Lecture 02: Asymptotic Analysis and Recurrences

Mohamed Alaa El-Dien Aly
Computer Engineering Department
Cairo University
Fall 2013

Agenda

- Asymptotic Notation
  - O-, Ω-, and Θ-notation
- Recurrences
  - Substitution Method
  - Recursion Tree
  - Master Theorem
- Divide-and-Conquer Examples

Acknowledgment
A lot of slides adapted from the slides of Erik Demaine and Charles Leiserson

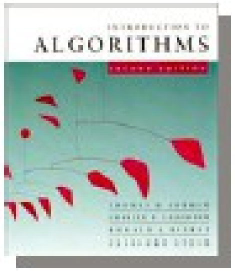


Asymptotic notation

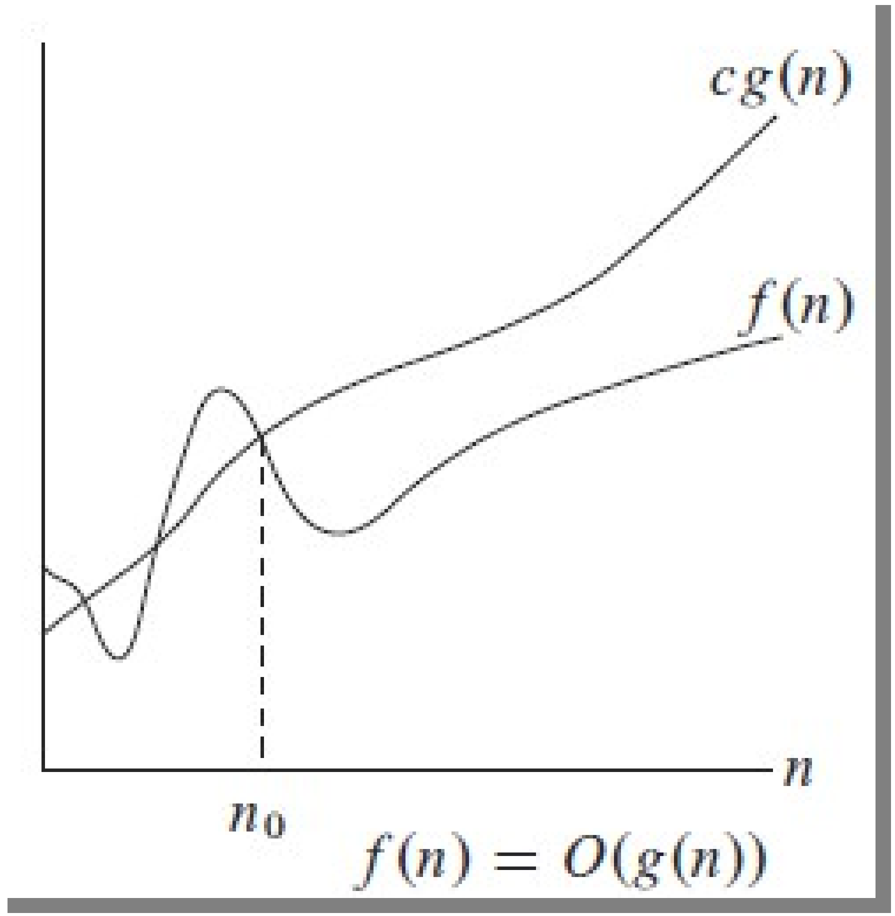
O-notation (upper bounds):

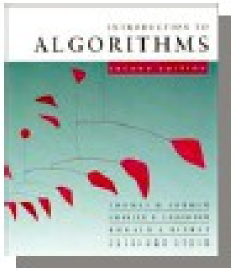
> We write $f(n) = O(g(n))$ if there exist constants $c > 0$, $n_0 > 0$ such that $0 \le f(n) \le cg(n)$ for all $n \ge n_0$.

EXAMPLE: $2n^2 = O(n^3)$ ($c = 1$, $n_0 = 2$)

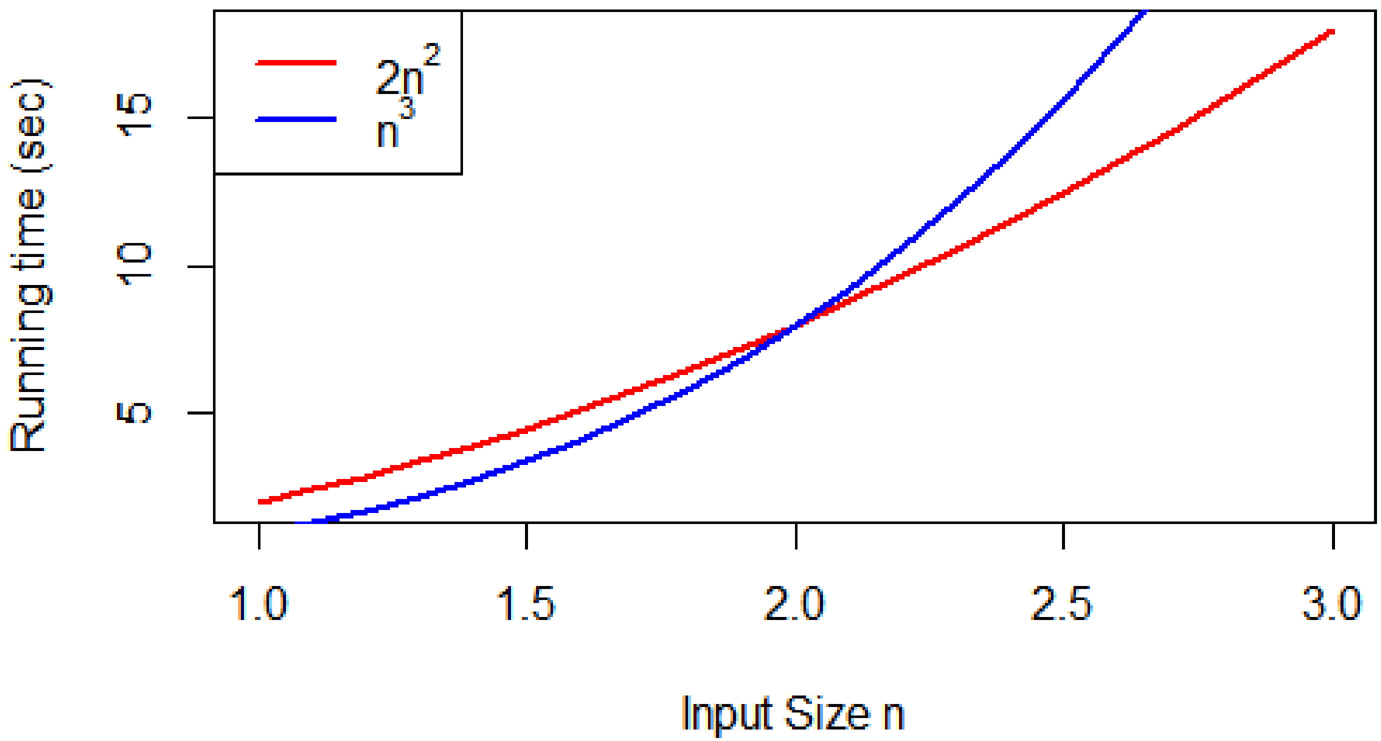


Asymptotic notation

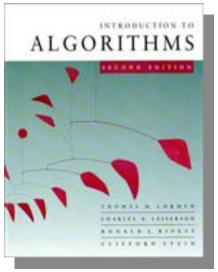




Asymptotic notation



EXAMPLE: $2n^2 = O(n^3) \qquad (c = 1, n_0 = 2)$



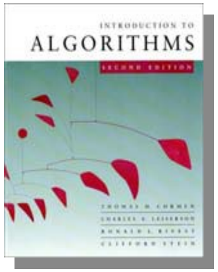
Asymptotic notation

O-notation (upper bounds):

We write $f(n) = O(g(n))$ if there exist constants $c > 0$, $n_0 > 0$ such that $0 \le f(n) \le cg(n)$ for all $n \ge n_0$.

EXAMPLE: $2n^2 = O(n^3)$ $\quad (c = 1, n_0 = 2)$

functions, not values



Asymptotic notation

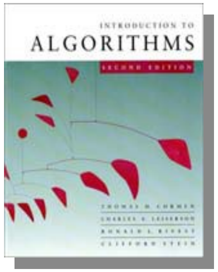
O-notation (upper bounds):

> We write $f(n) = O(g(n))$ if there exist constants $c > 0$, $n_0 > 0$ such that $0 \le f(n) \le cg(n)$ for all $n \ge n_0$.

Example: $2n^2 = O(n^3)$ $\quad (c = 1, n_0 = 2)$

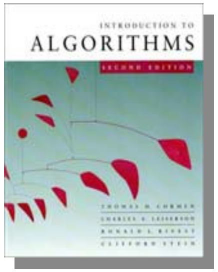
functions, not values

funny, "one-way" equality



Set definition of O-notation

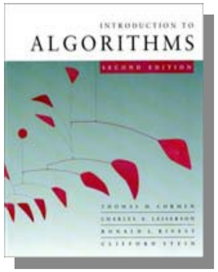
$O(g(n)) = \{ f(n)$: there exist constants $c > 0$, $n_0 > 0$ such that $0 \le f(n) \le cg(n)$ for all $n \ge n_0 \}$



Set definition of O-notation

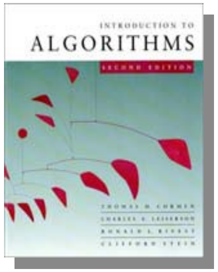
$O(g(n)) = \{ f(n) :$ there exist constants $c > 0, n_0 > 0$ such that $0 \le f(n) \le cg(n)$ for all $n \ge n_0 \}$

EXAMPLE: $2n^2 \in O(n^3)$



Macro substitution

Convention: A set in a formula represents an anonymous function in the set.



Macro substitution

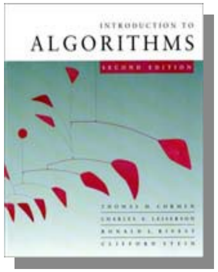
Convention: A set in a formula represents an anonymous function in the set.

EXAMPLE: $f(n) = n^3 + O(n^2)$

means

$f(n) = n^3 + h(n)$

for some $h(n) \in O(n^2)$.



Macro substitution

Convention: A set in a formula represents an anonymous function in the set.

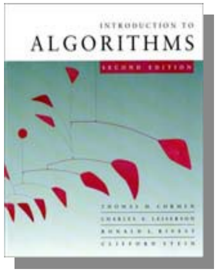
EXAMPLE: $n^2 + O(n) = O(n^2)$

means

for any $f(n) \in O(n)$:

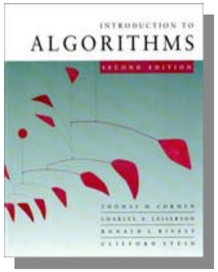
$$n^2 + f(n) = h(n)$$

for some $h(n) \in O(n^2)$.



Ω-notation (lower bounds)

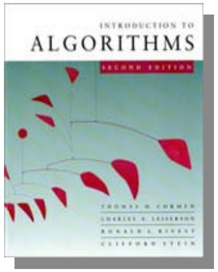
O-notation is an *upper-bound* notation. It makes no sense to say $f(n)$ is at least $O(n^2)$.



Ω-notation (lower bounds)

O-notation is an *upper-bound* notation. It makes no sense to say $f(n)$ is at least $O(n^2)$.

$\Omega(g(n)) = \{ f(n)$: there exist constants $c > 0$, $n_0 > 0$ such that $0 \leq cg(n) \leq f(n)$ for all $n \geq n_0 \}$



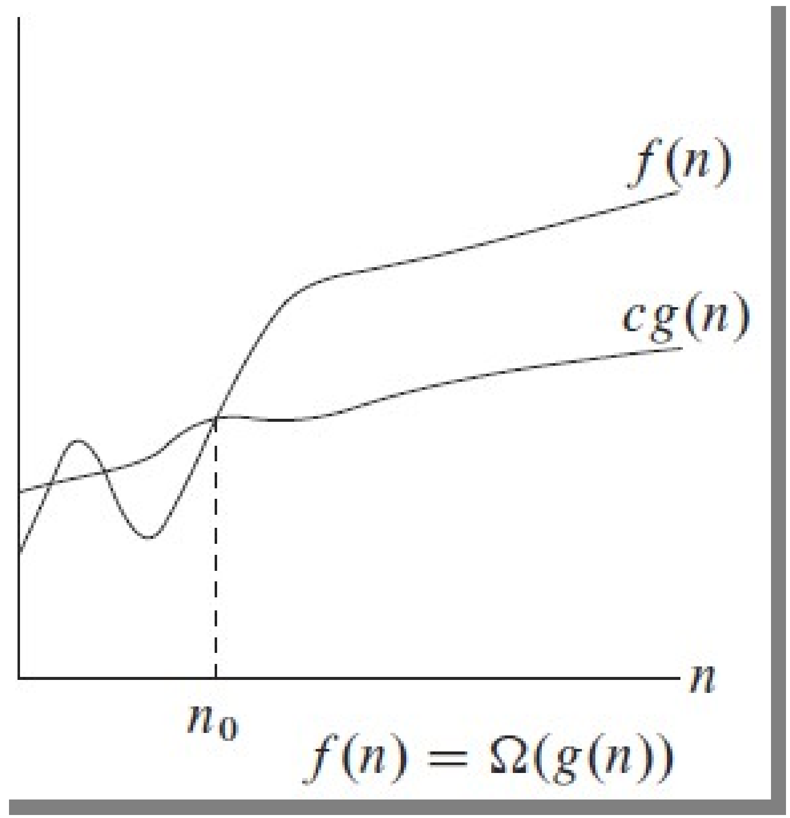
Ω-notation (lower bounds)

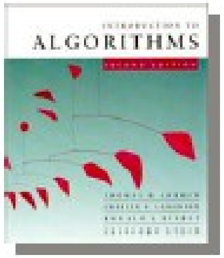
O-notation is an *upper-bound* notation. It makes no sense to say $f(n)$ is at least $O(n^2)$.

$\Omega(g(n)) = \{ f(n)$: there exist constants $c > 0$, $n_0 > 0$ such that $0 \le cg(n) \le f(n)$ for all $n \ge n_0 \}$

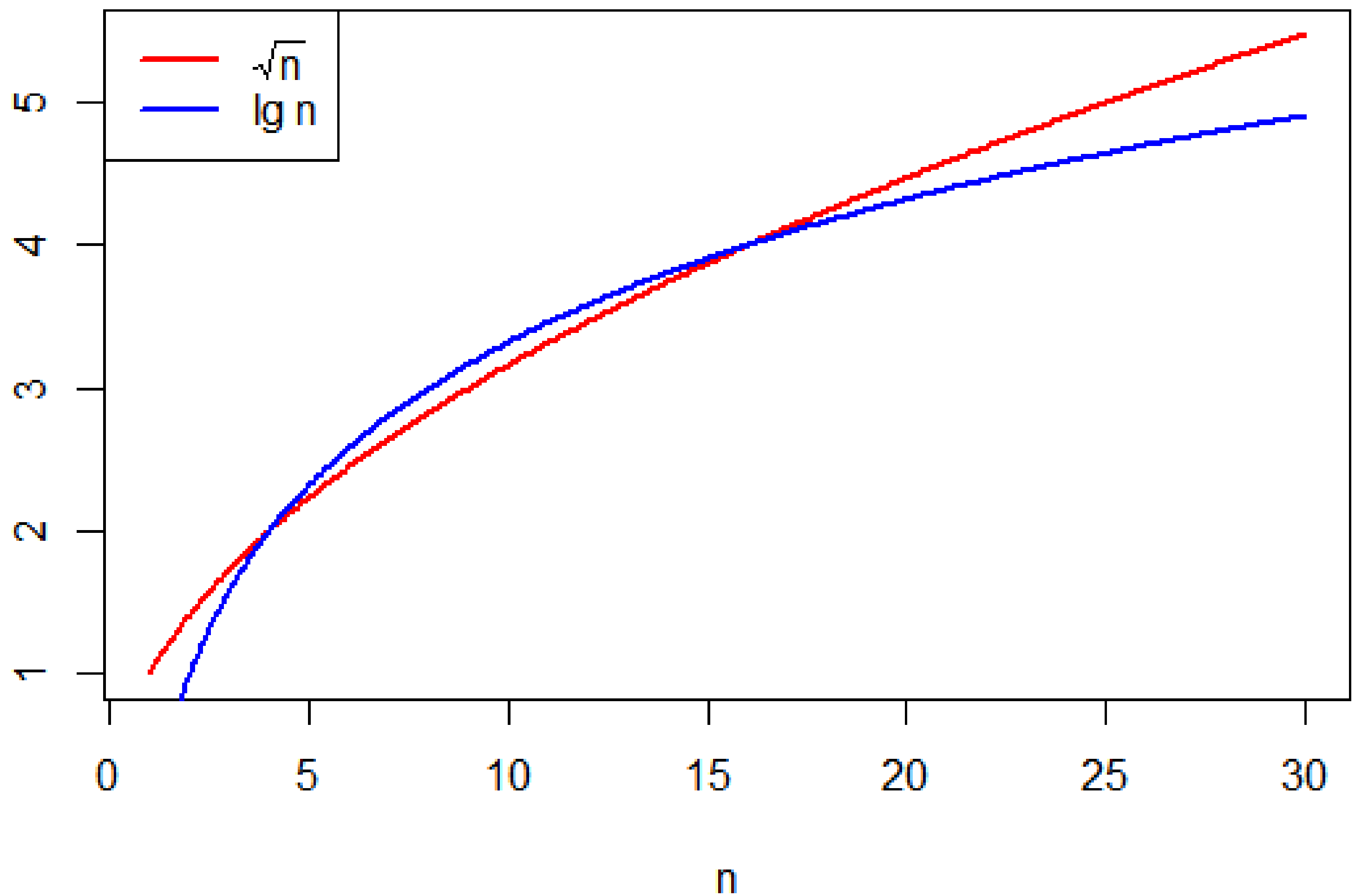
EXAMPLE: $\sqrt{n} = \Omega(\lg n)$ $(c = 1,\ n_0 = 16)$

Ω-notation (lower bounds)

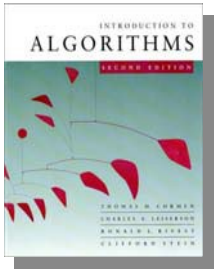




Ω-notation (lower bounds)

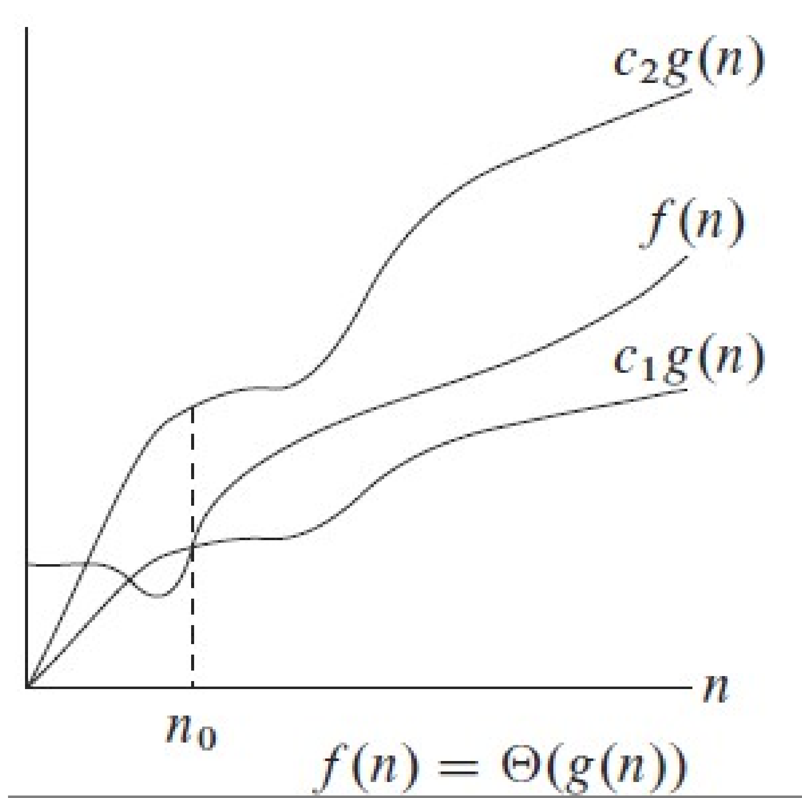


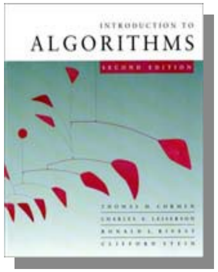
EXAMPLE: $\sqrt{n} = \Omega(\lg n) \quad (c = 1,\ n_0 = 16)$



Θ-notation (tight bounds)

$$\Theta(g(n)) = O(g(n)) \cap \Omega(g(n))$$





Θ-notation (tight bounds)

$$\Theta(g(n)) = O(g(n)) \cap \Omega(g(n))$$

EXAMPLE: $\frac{1}{2}n^2 - 2n = \Theta(n^2)$

Solving Recurrences

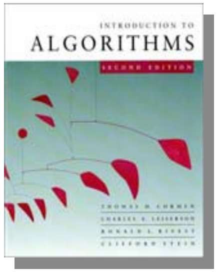
- For Merge Sort, we found that the running time was described by the recurrence

$$T(n) = 2T(n/2) + cn$$

which has a solution

$$T(n) = \Theta(n \lg n)$$

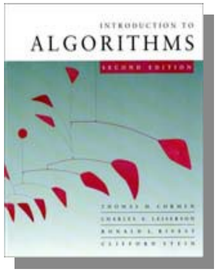
- In general, we need to solve recurrence to analyze Divide-and-Conquer algorithms



Substitution method

The most general method:

1. ***Guess*** the form of the solution.
2. ***Verify*** by induction.
3. ***Solve*** for constants.



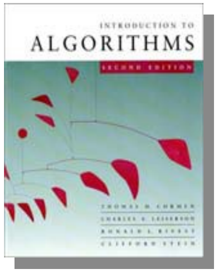
Substitution method

The most general method:

1. ***Guess*** the form of the solution.
2. ***Verify*** by induction.
3. ***Solve*** for constants.

EXAMPLE: $T(n) = 4T(n/2) + n$

- [Assume that $T(1) = \Theta(1)$.]
- Guess $O(n^3)$. (Prove O and Ω separately.)
- Assume that $T(k) \leq ck^3$ for $k < n$.
- Prove $T(n) \leq cn^3$ by induction.

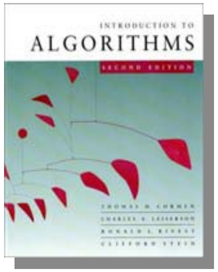


Example of substitution

$$
\begin{aligned}
T(n) &= 4T(n/2) + n \\
&\leq 4c(n/2)^3 + n \\
&= (c/2)n^3 + n \\
&= cn^3 - ((c/2)n^3 - n) \leftarrow \textit{desired} - \textit{residual} \\
&\leq cn^3 \leftarrow \textit{desired}
\end{aligned}
$$

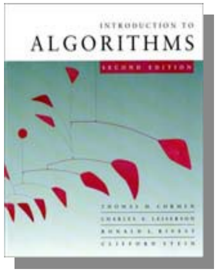
whenever $(c/2)n^3 - n \geq 0$, for example,
if $c \geq 2$ and $n \geq 1$.

residual



Example (continued)

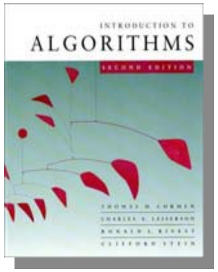
- We must also handle the initial conditions, that is, ground the induction with base cases.
- ***Base:*** $T(n) = \Theta(1)$ for all $n < n_0$, where n_0 is a suitable constant.
- For $1 \le n < n_0$, we have "$\Theta(1)$" $\le cn^3$, if we pick c big enough.



Example (continued)

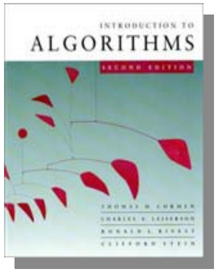
- We must also handle the initial conditions, that is, ground the induction with base cases.
- ***Base:*** $T(n) = \Theta(1)$ for all $n < n_0$, where n_0 is a suitable constant.
- For $1 \le n < n_0$, we have "$\Theta(1)$" $\le cn^3$, if we pick c big enough.

This bound is not tight!



A tighter upper bound?

We shall prove that $T(n) = O(n^2)$.

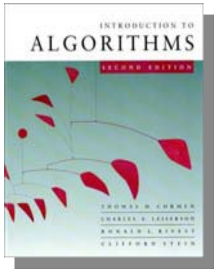


A tighter upper bound?

We shall prove that $T(n) = O(n^2)$.

Assume that $T(k) \le ck^2$ for $k < n$:

$$
\begin{aligned}
T(n) &= 4T(n/2) + n \\
&\le 4c(n/2)^2 + n \\
&= cn^2 + n \\
&= O(n^2)
\end{aligned}
$$



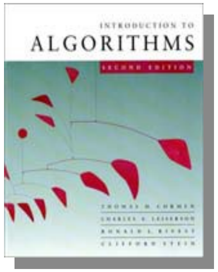
A tighter upper bound?

We shall prove that $T(n) = O(n^2)$.

Assume that $T(k) \le ck^2$ for $k < n$:

$$
\begin{aligned}
T(n) &= 4T(n/2) + n \\
&\le 4c(n/2)^2 + n \\
&= cn^2 + n \\
&= \cancel{O(n^2)}
\end{aligned}
$$

Wrong! We must prove the I.H.

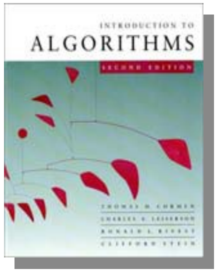


A tighter upper bound?

We shall prove that $T(n) = O(n^2)$.

Assume that $T(k) \le ck^2$ for $k < n$:

$$\begin{aligned}
T(n) &= 4T(n/2) + n \\
&\le 4c(n/2)^2 + n \\
&= cn^2 + n \\
&= \cancel{O(n^2)} \quad \textbf{\textit{Wrong!}} \text{ We must prove the I.H.} \\
&= cn^2 - (-n) \quad [\text{ desired} - \text{residual }] \\
&\le cn^2 \quad \text{for } \textbf{\textit{no}} \text{ choice of } c > 0. \text{ Lose!}
\end{aligned}$$

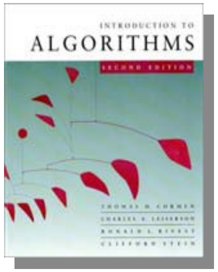


A tighter upper bound!

IDEA: Strengthen the inductive hypothesis.

- ***Subtract*** a low-order term.

Inductive hypothesis: $T(k) \le c_1 k^2 - c_2 k$ for $k < n$.



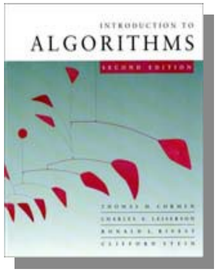
A tighter upper bound!

IDEA: Strengthen the inductive hypothesis.

- ***Subtract*** a low-order term.

Inductive hypothesis: $T(k) \le c_1 k^2 - c_2 k$ for $k < n$.

$$
\begin{aligned}
T(n) &= 4T(n/2) + n \\
&\le 4(c_1(n/2)^2 - c_2(n/2)) + n \\
&= c_1 n^2 - 2c_2 n + n \\
&= c_1 n^2 - c_2 n - (c_2 n - n) \\
&\le c_1 n^2 - c_2 n \ \text{ if } c_2 \ge 1.
\end{aligned}
$$



A tighter upper bound!

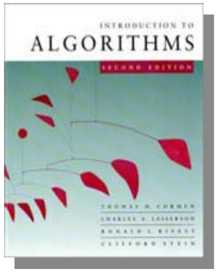
IDEA: Strengthen the inductive hypothesis.

- ***Subtract*** a low-order term.

Inductive hypothesis: $T(k) \le c_1k^2 - c_2k$ for $k < n$.

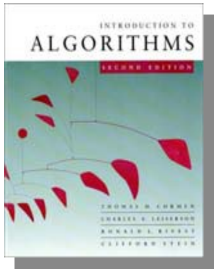
$$\begin{aligned}
T(n) &= 4T(n/2) + n \\
&\le 4(c_1(n/2)^2 - c_2(n/2)) + n \\
&= c_1n^2 - 2c_2n + n \\
&= c_1n^2 - c_2n - (c_2n - n) \\
&\le c_1n^2 - c_2n \ \text{ if } c_2 \ge 1.
\end{aligned}$$

Pick c_1 big enough to handle the initial conditions.



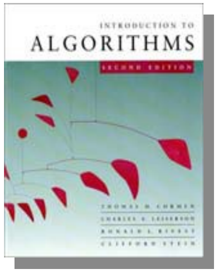
Recursion-tree method

- A recursion tree models the costs (time) of a recursive execution of an algorithm.
- The recursion-tree method can be unreliable, just like any method that uses ellipses (…).
- The recursion-tree method promotes intuition, however.
- The recursion tree method is good for generating guesses for the substitution method.



Example of recursion tree

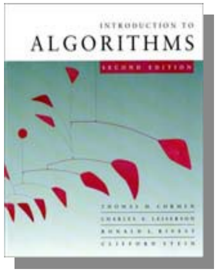
Solve $T(n) = T(n/4) + T(n/2) + n^2$:



Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:

$$T(n)$$

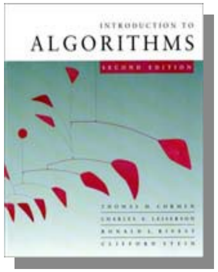


Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:

$$n^2$$

$$T(n/4) \qquad\qquad T(n/2)$$



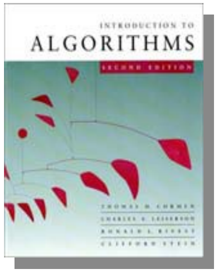
Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:

$$n^2$$

$$(n/4)^2 \qquad\qquad (n/2)^2$$

$$T(n/16) \qquad T(n/8) \qquad T(n/8) \qquad T(n/4)$$



Example of recursion tree

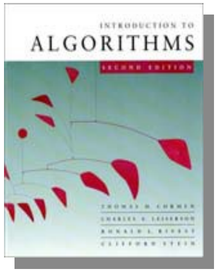
Solve $T(n) = T(n/4) + T(n/2) + n^2$:

$$n^2$$

$$(n/4)^2 \qquad (n/2)^2$$

$$(n/16)^2 \qquad (n/8)^2 \qquad (n/8)^2 \qquad (n/4)^2$$

$$\Theta(1)$$



Example of recursion tree

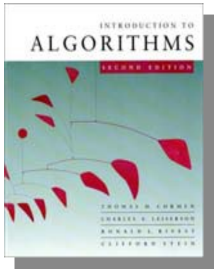
Solve $T(n) = T(n/4) + T(n/2) + n^2$:

$$n^2 \qquad\qquad n^2$$

$$(n/4)^2 \qquad (n/2)^2$$

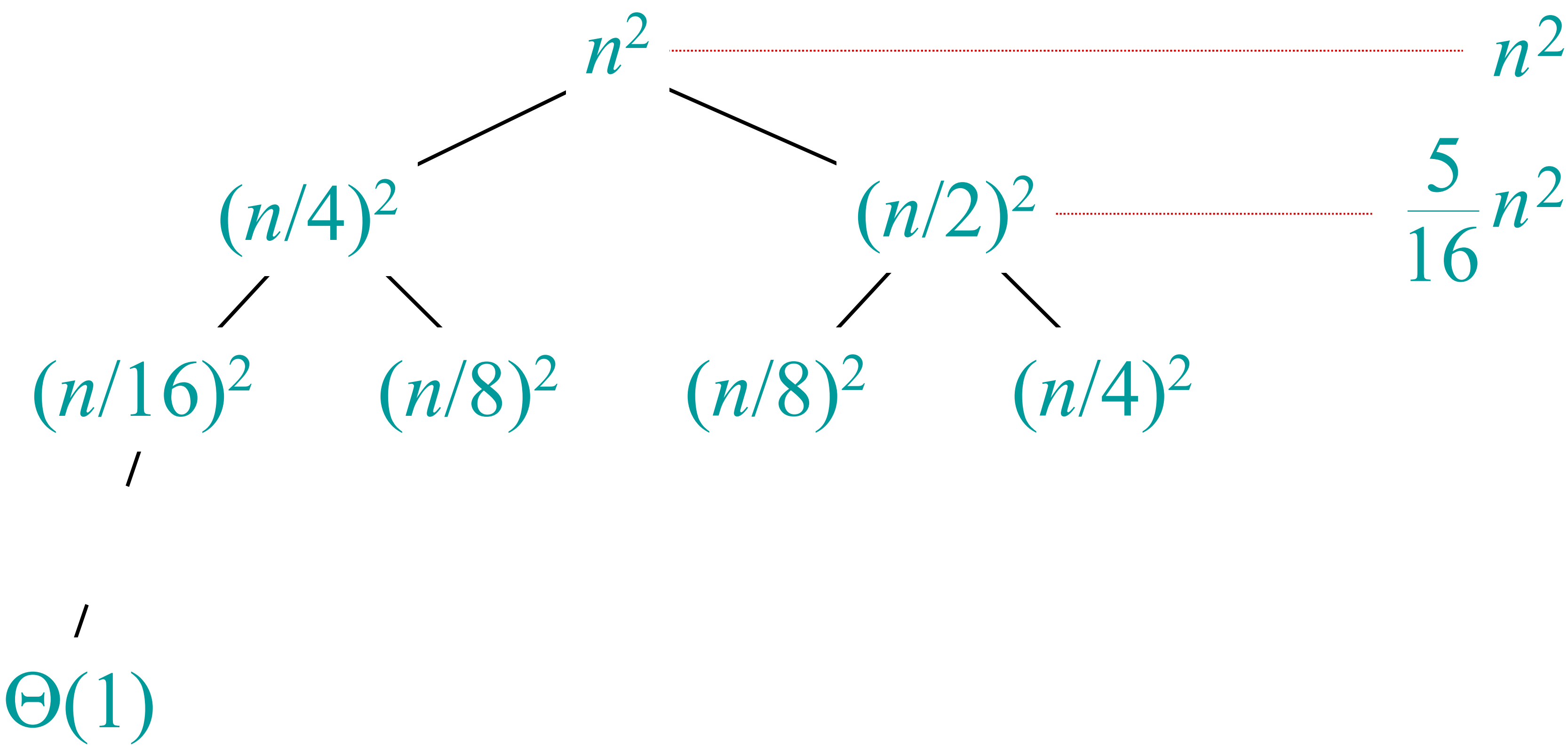
$$(n/16)^2 \quad (n/8)^2 \quad (n/8)^2 \quad (n/4)^2$$

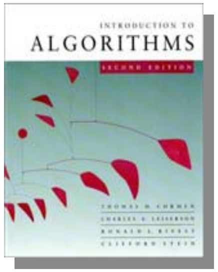
$$\Theta(1)$$



Example of recursion tree

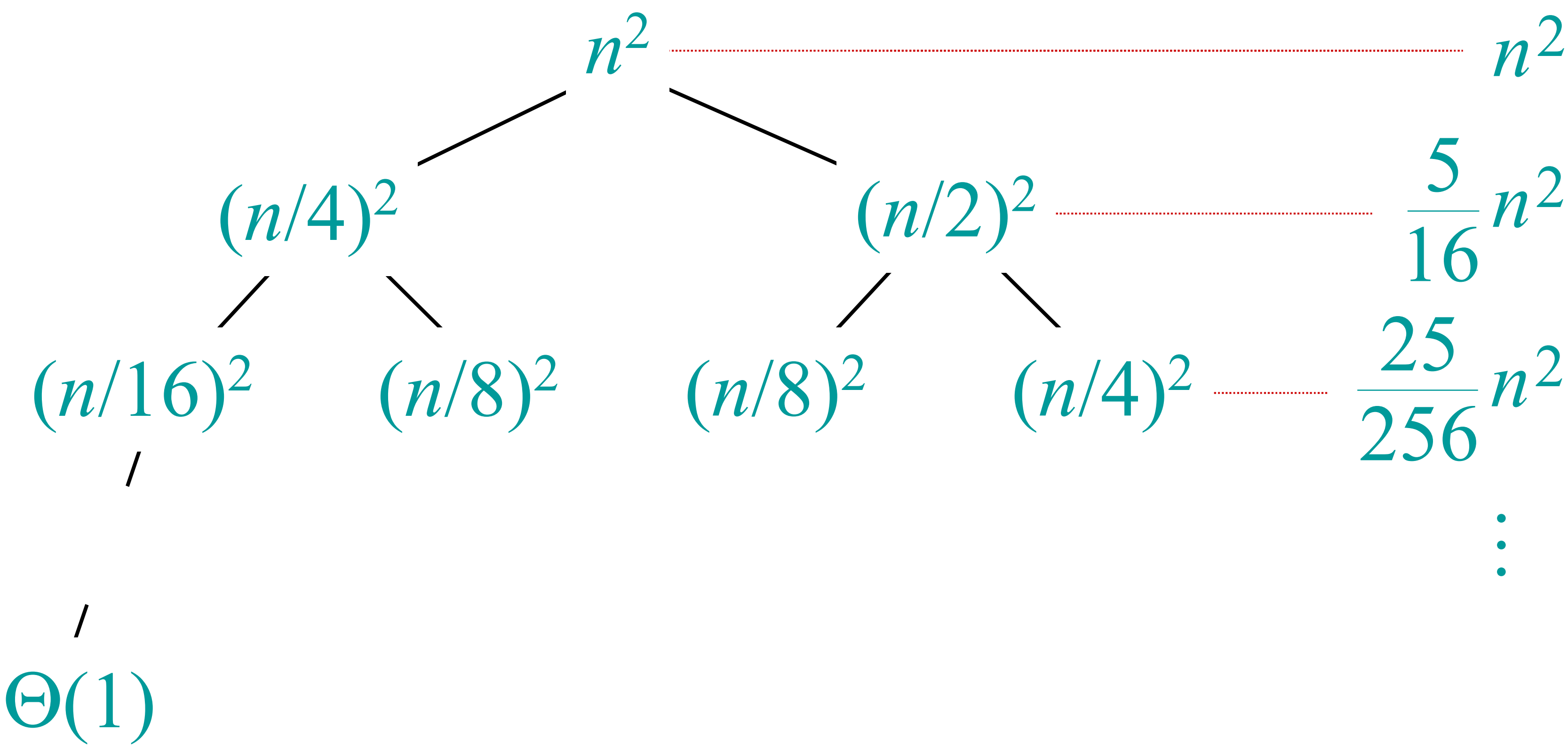
Solve $T(n) = T(n/4) + T(n/2) + n^2$:

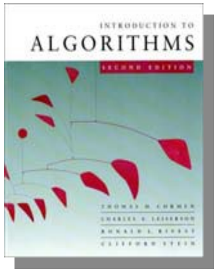




Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:





Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:

$$n^2 \quad \cdots \quad n^2$$

$$(n/4)^2 \qquad (n/2)^2 \quad \cdots \quad \frac{5}{16}n^2$$

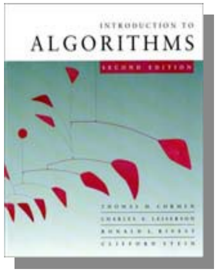
$$(n/16)^2 \qquad (n/8)^2 \qquad (n/8)^2 \qquad (n/4)^2 \quad \cdots \quad \frac{25}{256}n^2$$

$$\vdots$$

$$\Theta(1)$$

Total $= n^2\left(1 + \frac{5}{16} + \left(\frac{5}{16}\right)^2 + \left(\frac{5}{16}\right)^3 + \cdots\right)$

$= \Theta(n^2)$

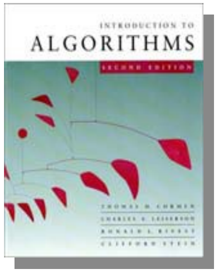


The master method

The master method applies to recurrences of the form

$$T(n) = a\,T(n/b) + f(n) \,,$$

where $a \geq 1$, $b > 1$, and f is asymptotically positive.

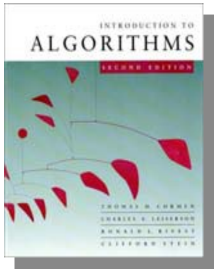


Three common cases

Compare $f(n)$ with $n^{\log_b a}$:

1. $f(n) = O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$.
   - $f(n)$ grows polynomially slower than $n^{\log_b a}$ (by an n^{ε} factor).

Solution: $T(n) = \Theta(n^{\log_b a})$.



Three common cases

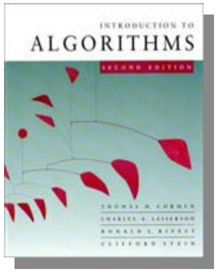
Compare $f(n)$ with $n^{\log_b a}$:

1. $f(n) = O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$.
   - $f(n)$ grows polynomially slower than $n^{\log_b a}$ (by an n^{ε} factor).

   Solution: $T(n) = \Theta(n^{\log_b a})$.

2. $f(n) = \Theta(n^{\log_b a} \lg^k n)$ for some constant $k \geq 0$.
   - $f(n)$ and $n^{\log_b a}$ grow at similar rates.

   Solution: $T(n) = \Theta(n^{\log_b a} \lg^{k+1} n)$.



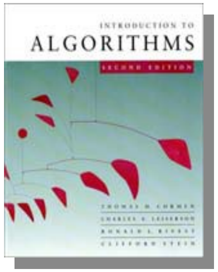
Three common cases (cont.)

Compare $f(n)$ with $n^{\log_b a}$:

3. $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some constant $\varepsilon > 0$.
   - $f(n)$ grows polynomially faster than $n^{\log_b a}$ (by an n^{ε} factor),

   and $f(n)$ satisfies the ***regularity condition*** that $a f(n/b) \leq c f(n)$ for some constant $c < 1$.

   Solution: $T(n) = \Theta(f(n))$.



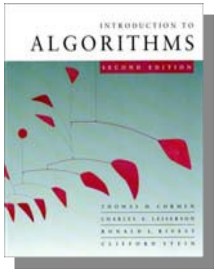
Examples

Ex. $T(n) = 4T(n/2) + n$

$a = 4,\ b = 2 \Rightarrow n^{\log_b a} = n^2;\ f(n) = n.$

CASE 1: $f(n) = O(n^{2-\varepsilon})$ for $\varepsilon = 1$.

$\therefore\ T(n) = \Theta(n^2)$.



Examples

Ex. $T(n) = 4T(n/2) + n$

$a = 4, b = 2 \Rightarrow n^{\log_b a} = n^2; f(n) = n.$

CASE 1: $f(n) = O(n^{2-\varepsilon})$ for $\varepsilon = 1$.

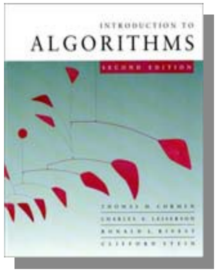
$\therefore\ T(n) = \Theta(n^2).$

Ex. $T(n) = 4T(n/2) + n^2$

$a = 4, b = 2 \Rightarrow n^{\log_b a} = n^2; f(n) = n^2.$

CASE 2: $f(n) = \Theta(n^2 \lg^0 n)$, that is, $k = 0$.

$\therefore\ T(n) = \Theta(n^2 \lg n).$



Examples

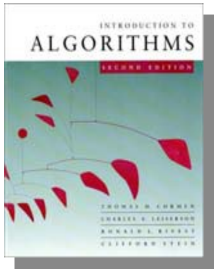
Ex. $T(n) = 4T(n/2) + n^3$

$a = 4,\ b = 2 \Rightarrow n^{\log_b a} = n^2;\ f(n) = n^3.$

CASE 3: $f(n) = \Omega(n^{2+\varepsilon})$ for $\varepsilon = 1$

and $4(n/2)^3 \le cn^3$ (reg. cond.) for $c = 1/2$.

$\therefore\ T(n) = \Theta(n^3)$.



Examples

Ex. $T(n) = 4T(n/2) + n^3$

$a = 4,\ b = 2 \Rightarrow n^{\log_b a} = n^2;\ f(n) = n^3.$

Case 3: $f(n) = \Omega(n^{2+\varepsilon})$ for $\varepsilon = 1$

and $4(n/2)^3 \le cn^3$ (reg. cond.) for $c = 1/2$.

$\therefore\ T(n) = \Theta(n^3)$.

Ex. $T(n) = 4T(n/2) + n^2/\lg n$

$a = 4,\ b = 2 \Rightarrow n^{\log_b a} = n^2;\ f(n) = n^2/\lg n.$

Master method does not apply. In particular, for every constant $\varepsilon > 0$, we have $n^{\varepsilon} = \omega(\lg n)$.

Intuition of Master Theorem

$$f(n) = cn$$

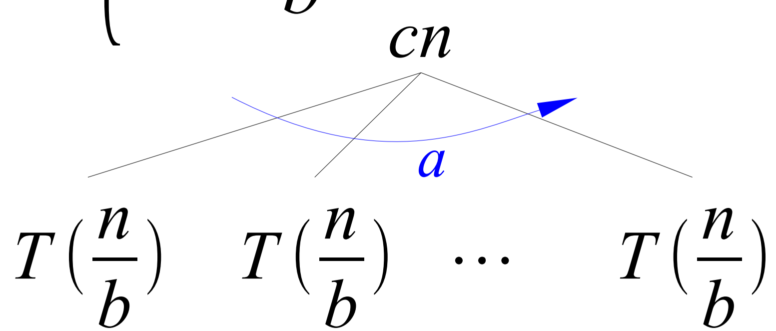
$$T(n) = \begin{cases} c & n = 1 \\ aT\left(\dfrac{n}{b}\right) + cn & n > 1 \end{cases}$$

$$T(n)$$

Intuition of Master Theorem

$$f(n) = cn$$

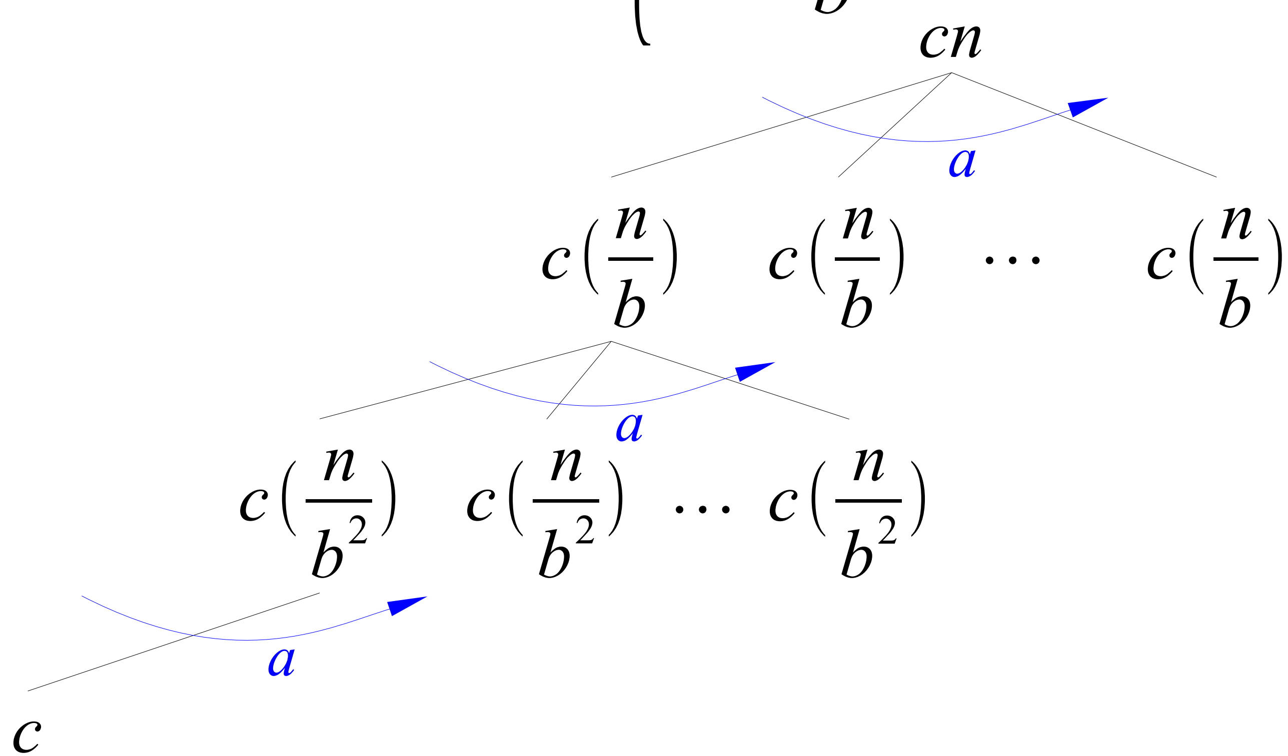
$$T(n) = \begin{cases} c & n = 1 \\ aT\left(\frac{n}{b}\right) + cn & n > 1 \end{cases}$$



Intuition of Master Theorem

$$f(n) = cn$$

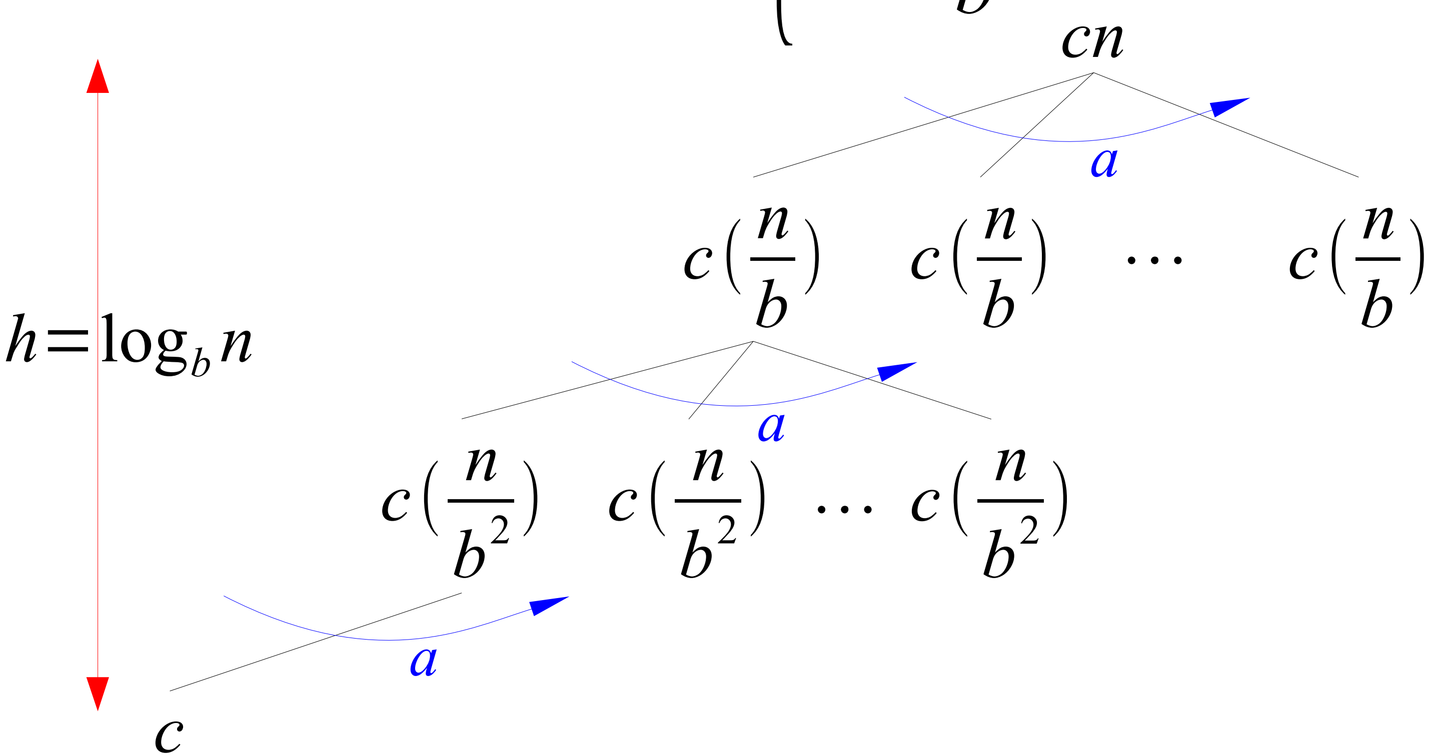
$$T(n) = \begin{cases} c & n = 1 \\ aT\left(\frac{n}{b}\right) + cn & n > 1 \end{cases}$$



Intuition of Master Theorem

$$f(n) = cn$$

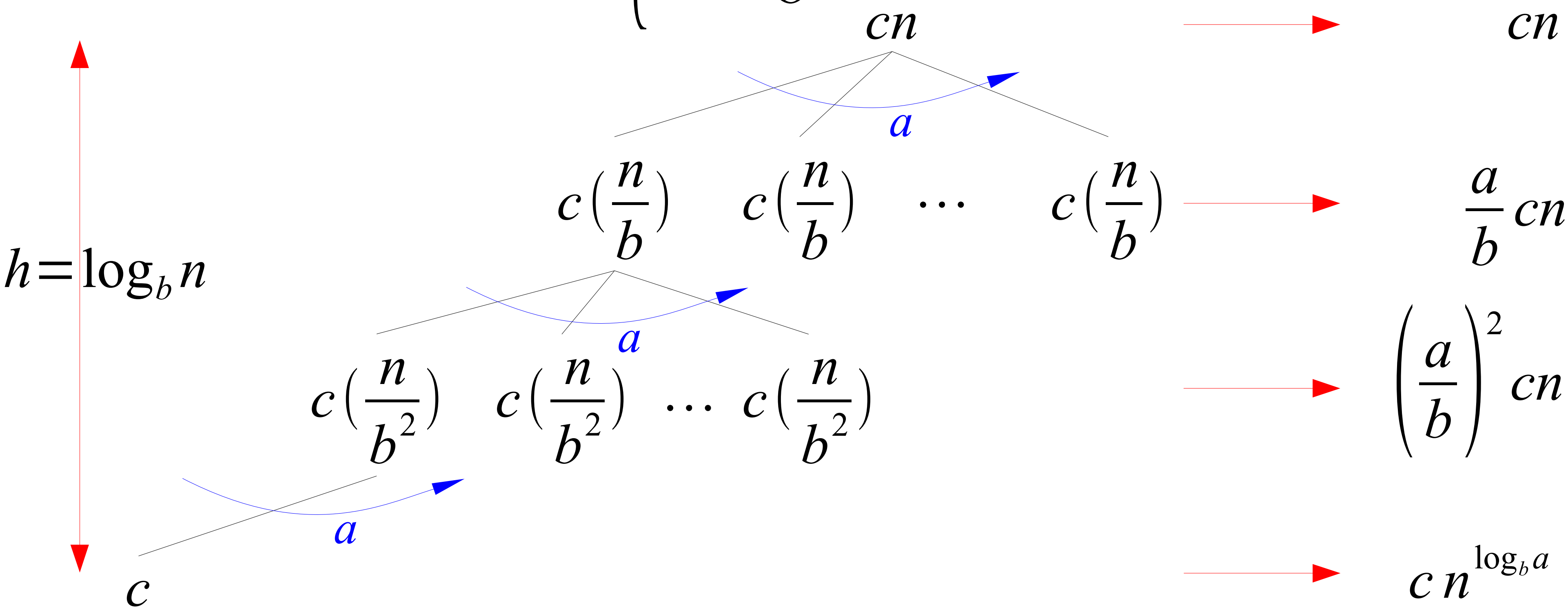
$$T(n) = \begin{cases} c & n = 1 \\ aT\left(\frac{n}{b}\right) + cn & n > 1 \end{cases}$$



Intuition of Master Theorem

$$f(n) = cn$$

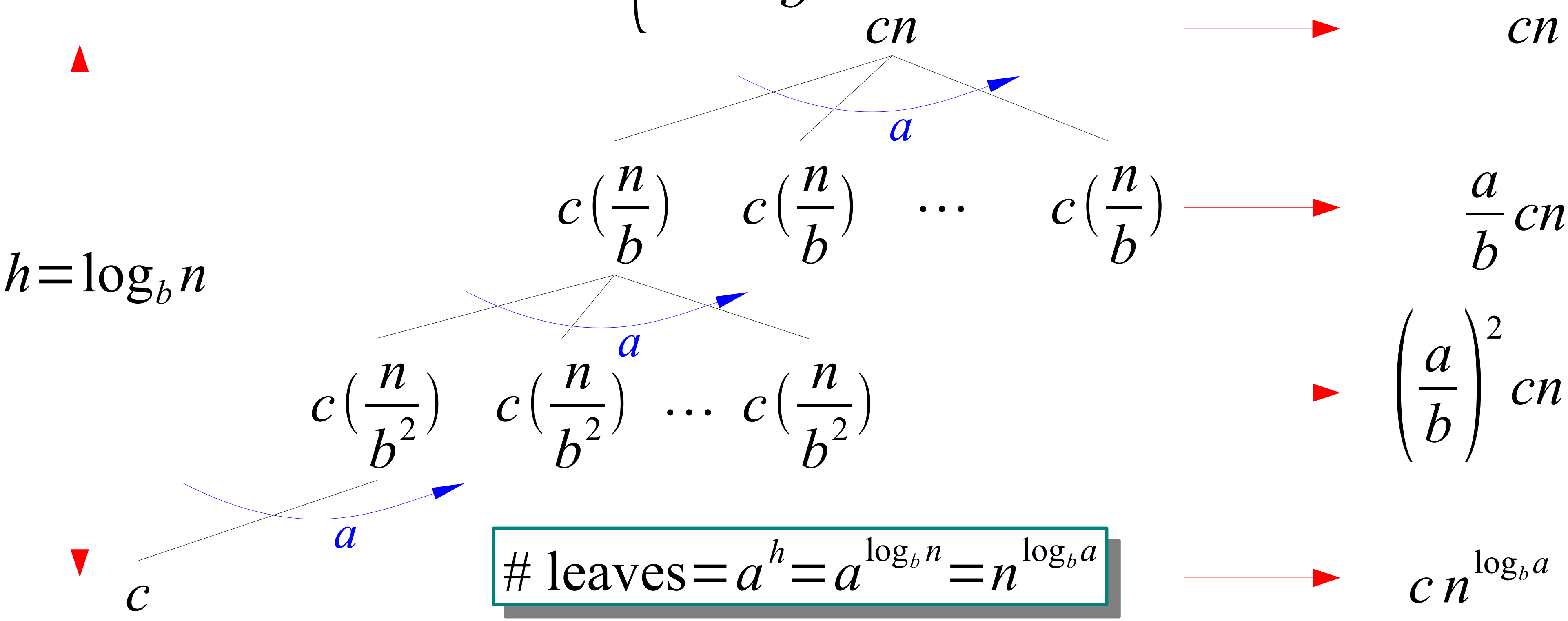
$$T(n) = \begin{cases} c & n = 1 \\ aT\left(\frac{n}{b}\right) + cn & n > 1 \end{cases}$$



Intuition of Master Theorem

$$f(n) = cn$$

$$T(n) = \begin{cases} c & n = 1 \\ aT\left(\frac{n}{b}\right) + cn & n > 1 \end{cases}$$

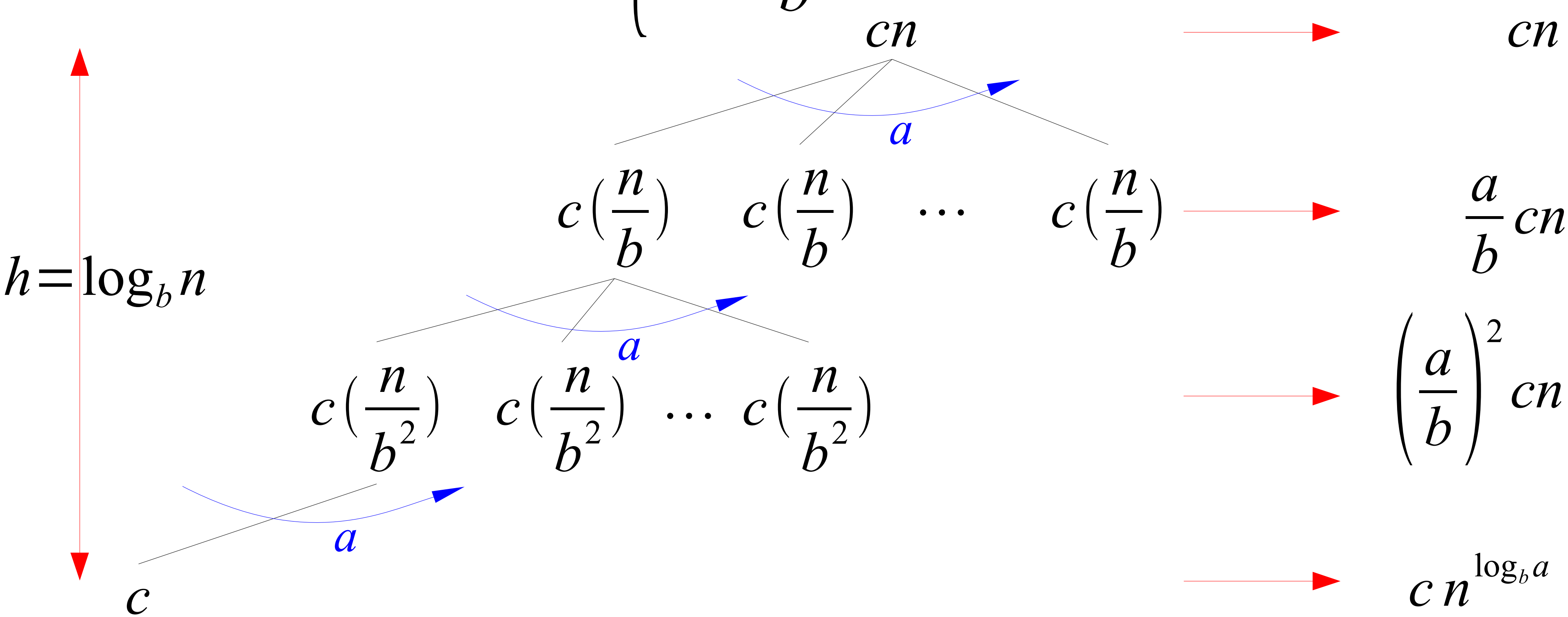


How? $\quad a^{\log_b n} = b^{\log_b \left(a^{\log_b n}\right)} = b^{\log_b n \log_b a} = \left(b^{\log_b n}\right)^{\log_b a} = n^{\log_b a}$

Intuition of Master Theorem

$$f(n) = cn$$

$$T(n) = \begin{cases} c & n = 1 \\ aT\left(\frac{n}{b}\right) + cn & n > 1 \end{cases}$$

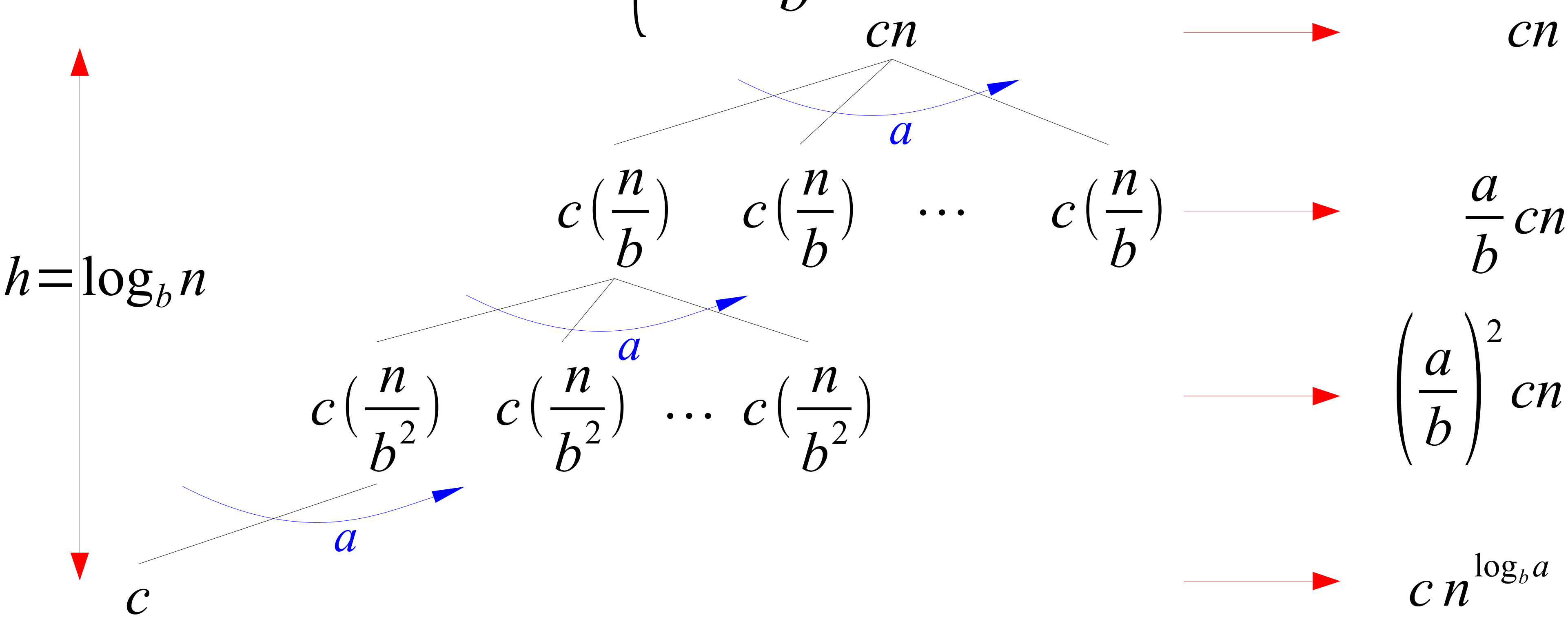


$$T(n) = cn \sum_{k=0}^{\log_b n} \left(\frac{a}{b}\right)^k = c\,n \frac{1 - (a/b)^{\log_b n + 1}}{1 - a/b}$$

Intuition of Master Theorem

$$f(n) = cn$$

$$T(n) = \begin{cases} c & n = 1 \\ aT\left(\frac{n}{b}\right) + cn & n > 1 \end{cases}$$



Case 1: a > b

Weights are increasing downwards

$$T(n) = c\,n \frac{1 - (a/b)^{\log_b n + 1}}{1 - a/b} = \Theta\left(n^{\log_b a}\right)$$

Intuition of Master Theorem

$$f(n)=cn$$

$$T(n)=\begin{cases} c & n=1 \\ aT\left(\frac{n}{b}\right)+cn & n>1 \end{cases}$$

$$cn \longrightarrow cn$$

$$a$$

$$c\left(\frac{n}{b}\right) \quad c\left(\frac{n}{b}\right) \quad \cdots \quad c\left(\frac{n}{b}\right) \longrightarrow \frac{a}{b}cn$$

$$a$$

$$c\left(\frac{n}{b^2}\right) \quad c\left(\frac{n}{b^2}\right) \quad \cdots \quad c\left(\frac{n}{b^2}\right) \longrightarrow \left(\frac{a}{b}\right)^2 cn$$

$$a$$

$$c \longrightarrow c\,n^{\log_b a}$$

$$h=\log_b n$$

Case 2: a = b

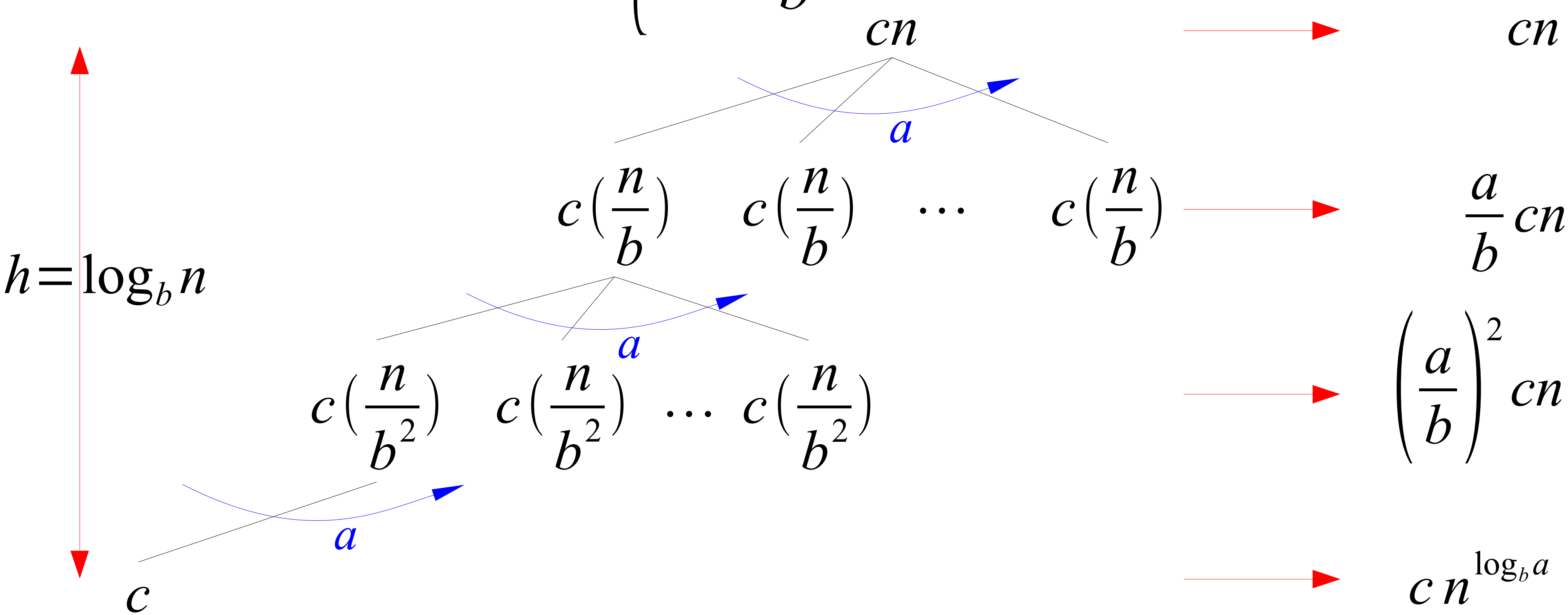
Weights are equal across levels

$$T(n)=c\,n\frac{1-(a/b)^{\log_b n+1}}{1-a/b}=\Theta(n\log_b n)$$

Intuition of Master Theorem

$$f(n)=cn$$

$$T(n)=\begin{cases} c & n=1 \\ aT\left(\frac{n}{b}\right)+cn & n>1 \end{cases}$$



Case 3: a < b

Weights are decreasing downwards

$$T(n)=c\,n\frac{1-(a/b)^{\log_b n+1}}{1-a/b}=\Theta(n)$$

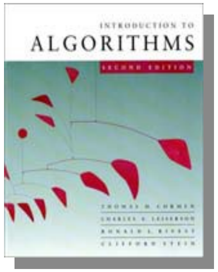
Intuition of Master Theorem

Summary

$$f(n) = cn$$

$$T(n) = \begin{cases} c & n = 1 \\ aT\left(\frac{n}{b}\right) + cn & n > 1 \end{cases}$$

$$T(n) = \begin{cases} \Theta(n) & , a < b \\ \Theta(n \log_b n) & , a = b \\ \Theta\left(n^{\log_b a}\right) & , a > b \end{cases}$$



Idea of master theorem

Recursion tree:

$f(n)$ .... $f(n)$

a

$f(n/b)$ $f(n/b)$ $\cdots$ $f(n/b)$ .... $af(n/b)$

a

$h = \log_b n$

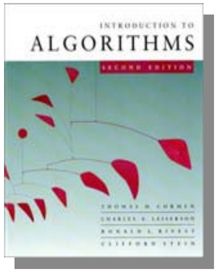
$f(n/b^2)$ $f(n/b^2)$ $\cdots$ $f(n/b^2)$ .... $a^2 f(n/b^2)$

$\vdots$ $\vdots$

T(1) .... $n^{\log_b a}$ T(1)

$\Theta(n^{\log_b a})$

CASE 1: The weight increases geometrically from the root to the leaves. The leaves hold a constant fraction of the total weight.



Idea of master theorem

Recursion tree:

$h = \log_b n$

$f(n)$ .... $f(n)$

a

$f(n/b)\quad f(n/b)\quad \cdots\quad f(n/b)$ .... $a\,f(n/b)$

a

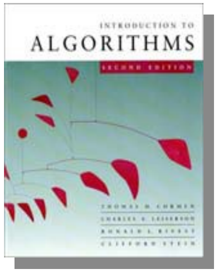
$f(n/b^2)\quad f(n/b^2)\quad \cdots\quad f(n/b^2)$ .... $a^2 f(n/b^2)$

$\vdots$

$T(1)$ .... $n^{\log_b a}\ T(1)$

CASE 2: ($k = 0$) The weight is approximately the same on each of the $\log_b n$ levels.

$\Theta(n^{\log_b a} \lg n)$



Idea of master theorem

Recursion tree:

$f(n)$ …… $f(n)$

a

$f(n/b)$ $f(n/b)$ $\cdots$ $f(n/b)$ …… $a f(n/b)$

a

$h = \log_b n$

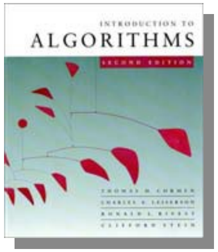
$f(n/b^2)$ $f(n/b^2)$ $\cdots$ $f(n/b^2)$ …… $a^2 f(n/b^2)$

$\vdots$

$T(1)$ …… $n^{\log_b a}$ (1)

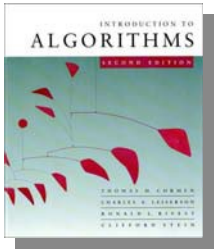
$\Theta(f(n))$

CASE 3: The weight decreases geometrically from the root to the leaves. The root holds a constant fraction of the total weight.



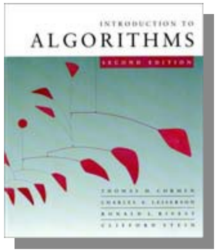
The divide-and-conquer design paradigm

1. ***Divide*** the problem (instance) into subproblems.
2. ***Conquer*** the subproblems by solving them recursively.
3. ***Combine*** subproblem solutions.



Merge sort

1. ***Divide:*** Trivial.
2. ***Conquer:*** Recursively sort 2 subarrays.
3. ***Combine:*** Linear-time merge.



Merge sort

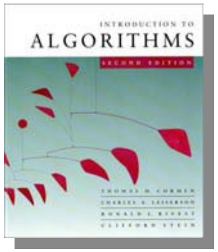
1. ***Divide:*** Trivial.
2. ***Conquer:*** Recursively sort 2 subarrays.
3. ***Combine:*** Linear-time merge.

$$T(n) = 2\,T(n/2) + \Theta(n)$$

 # subproblems

subproblem size

work dividing and combining



Master theorem (reprise)

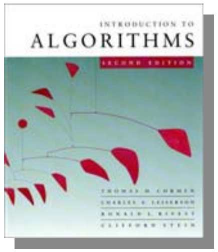
$$T(n) = a\,T(n/b) + f(n)$$

CASE 1: $f(n) = O(n^{\log_b a - \varepsilon})$, constant $\varepsilon > 0$
$\Rightarrow T(n) = \Theta(n^{\log_b a})$.

CASE 2: $f(n) = \Theta(n^{\log_b a} \lg^k n)$, constant $k \geq 0$
$\Rightarrow T(n) = \Theta(n^{\log_b a} \lg^{k+1} n)$.

CASE 3: $f(n) = \Omega(n^{\log_b a + \varepsilon})$, constant $\varepsilon > 0$,
and regularity condition
$\Rightarrow T(n) = \Theta(f(n))$.

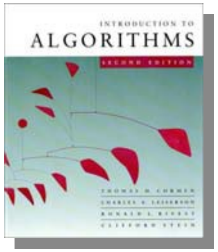
Merge sort: $a = 2$, $b = 2$ $\Rightarrow$ $n^{\log_b a} = n^{\log_2 2} = n$
$\Rightarrow$ CASE 2 $(k = 0)$ $\Rightarrow$ $T(n) = \Theta(n \lg n)$.



Binary search

Find an element in a sorted array:

1. ***Divide:*** Check middle element.
2. ***Conquer:*** Recursively search 1 subarray.
3. ***Combine:*** Trivial.



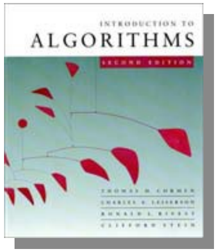
Binary search

Find an element in a sorted array:

1. ***Divide:*** Check middle element.
2. ***Conquer:*** Recursively search 1 subarray.
3. ***Combine:*** Trivial.

Example: Find 9

3	5	7	8	9	12	15



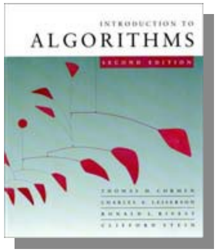
Binary search

Find an element in a sorted array:

1. ***Divide:*** Check middle element.
2. ***Conquer:*** Recursively search 1 subarray.
3. ***Combine:*** Trivial.

Example: Find 9

3 5 7 8 9 12 15



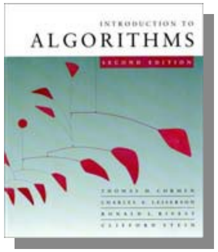
Binary search

Find an element in a sorted array:

1. ***Divide:*** Check middle element.
2. ***Conquer:*** Recursively search 1 subarray.
3. ***Combine:*** Trivial.

Example: Find 9

3 5 7 8 9 12 15



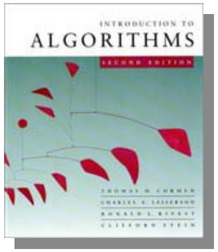
Binary search

Find an element in a sorted array:

1. ***Divide:*** Check middle element.
2. ***Conquer:*** Recursively search 1 subarray.
3. ***Combine:*** Trivial.

Example: Find 9

3 5 7 8 9 12 15



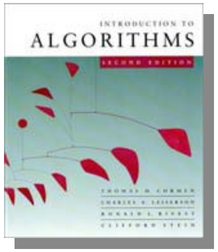
Binary search

Find an element in a sorted array:

1. ***Divide:*** Check middle element.
2. ***Conquer:*** Recursively search 1 subarray.
3. ***Combine:*** Trivial.

Example: Find 9

3 5 7 8 9 12 15



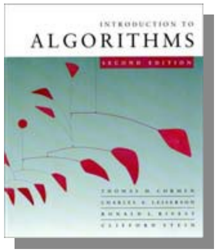
Binary search

Find an element in a sorted array:

1. ***Divide:*** Check middle element.
2. ***Conquer:*** Recursively search 1 subarray.
3. ***Combine:*** Trivial.

Example: Find 9

3 5 7 8 9 12 15



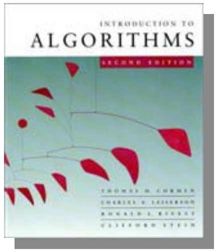
Recurrence for binary search

$$T(n) = 1\, T(n/2) + \Theta(1)$$

 # subproblems

subproblem size

work dividing and combining



Recurrence for binary search

$$T(n) = 1\,T(n/2) + \Theta(1)$$

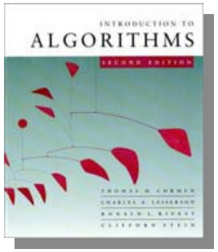
 # subproblems

subproblem size

work dividing and combining

$$n^{\log_b a} = n^{\log_2 1} = n^0 = 1 \Rightarrow \text{CASE 2 } (k = 0)$$

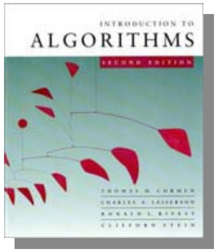
$$\Rightarrow T(n) = \Theta(\lg n)\,.$$



Powering a number

Problem: Compute a^n, where $n \in \mathbb{N}$.

Naive algorithm: $\Theta(n)$.



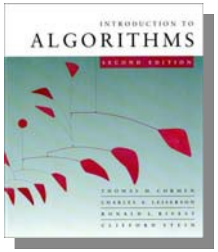
Powering a number

Problem: Compute a^n, where $n \in \mathbb{N}$.

Naive algorithm: $\Theta(n)$.

Divide-and-conquer algorithm:

$$a^n = \begin{cases} a^{n/2} \cdot a^{n/2} & \text{if } n \text{ is even;} \\ a^{(n-1)/2} \cdot a^{(n-1)/2} \cdot a & \text{if } n \text{ is odd.} \end{cases}$$



Powering a number

Problem: Compute a^n, where $n \in \mathbb{N}$.

Naive algorithm: $\Theta(n)$.

Divide-and-conquer algorithm:

$$a^n = \begin{cases} a^{n/2} \cdot a^{n/2} & \text{if } n \text{ is even;} \\ a^{(n-1)/2} \cdot a^{(n-1)/2} \cdot a & \text{if } n \text{ is odd.} \end{cases}$$

$$T(n) = T(n/2) + \Theta(1) \Rightarrow T(n) = \Theta(\lg n) .$$

Recap

- Asymptotic Notation
  - O-, Ω-, and Θ-notation
- Recurrences
  - Substitution Method
  - Recursion Tree
  - Master Theorem
- Divide-and-Conquer Examples
- Next:
  - Heapsort and Quicksort