

CMP461: Algorithms



Lecture 10: Minimal Spanning Trees

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Cairo University
Fall 2013

Agenda

- Greedy Algorithm Properties
- Graphs
 - Representation
 - Minimal Spanning Trees

Acknowledgment

A lot of slides adapted from the slides of Erik Demaine, Charles Leiserson, and David Luebke.

Greedy Algorithms

Related to Dynamic Programming, depends on two properties:

1. Optimal substructure

The optimal solution to **the main problem** contains optimal solutions to **subproblems** (**like** DP).

2. Greedy choice

At each step, we only need to solve **one subproblem** and make one choice, the **greedy choice** i.e. the best choice at the moment (**unlike** DP).

Finding the locally optimal solutions will get us to the global optimum.



Graphs (review)

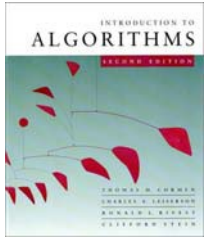
Definition. A *directed graph (digraph)* $G = (V, E)$ is an ordered pair consisting of

- a set V of *vertices* (singular: *vertex*),
- a set $E \subseteq V \times V$ of *edges*.

In an *undirected graph* $G = (V, E)$, the edge set E consists of *unordered* pairs of vertices.

In either case, we have $|E| = O(V^2)$. Moreover, if G is connected, then $|E| \geq |V| - 1$, which implies that $\lg |E| = \Theta(\lg V)$.

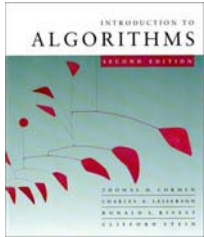
(Review CLRS, Appendix B.)



Adjacency-matrix representation

The *adjacency matrix* of a graph $G = (V, E)$, where $V = \{1, 2, \dots, n\}$, is the matrix $A[1..n, 1..n]$ given by

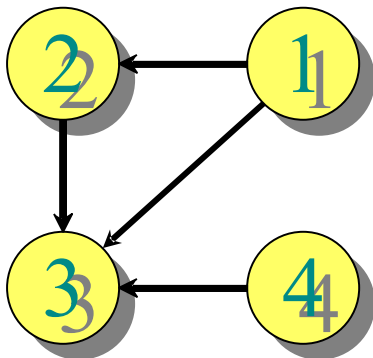
$$A[i, j] = \begin{cases} 1 & \text{if } (i, j) \in E, \\ 0 & \text{if } (i, j) \notin E. \end{cases}$$



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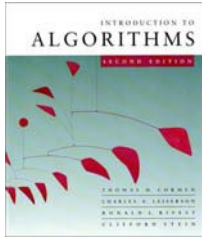
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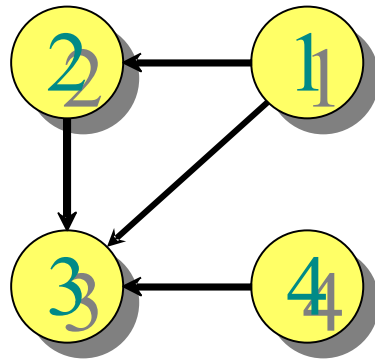
A	1	2	3	4
1	0	1	1	0
2	0	0	1	0
3	0	0	0	0
4	0	0	1	0

$\Theta(V^2)$ storage
 \Rightarrow *dense*
representation.



Adjacency-list representation

An *adjacency list* of a vertex $v \in V$ is the list $Adj[v]$ of vertices adjacent to v .

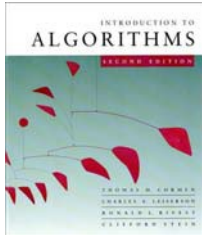


$$Adj[1] = \{2, 3\}$$

$$Adj[2] = \{3\}$$

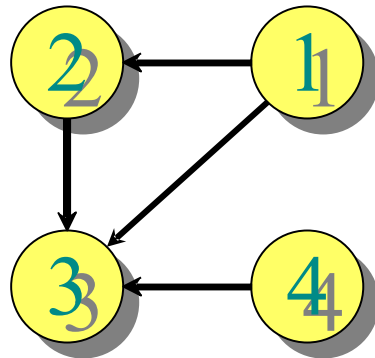
$$Adj[3] = \{\}$$

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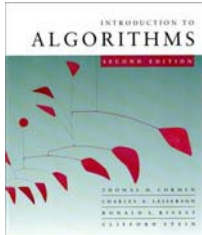
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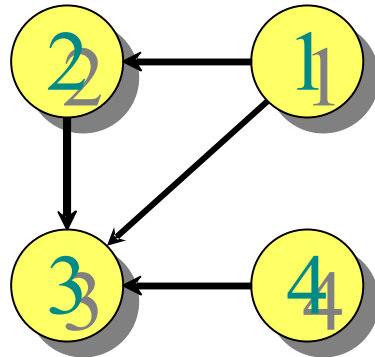
For undirected graphs, $|Adj[v]| = degree(v)$.

For digraphs, $|Adj[v]| = out-degree(v)$.



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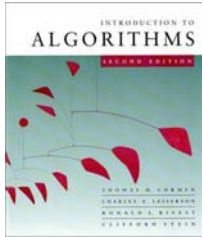
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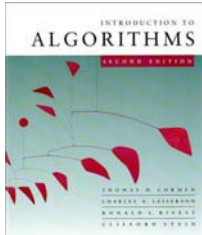
Handshaking Lemma: $\sum_{v \in V} |Adj[v]| = 2 |E|$ for undirected graphs
i.e. adjacency lists use $\Theta(V + E)$ storage \rightarrow a **sparse** representation
for either type of graph



Minimum spanning trees

Input: A connected, undirected graph $G = (V, E)$ with weight function $w : E \rightarrow \mathbb{R}$.

- For simplicity, assume that all edge weights are distinct. (CLRS covers the general case.)



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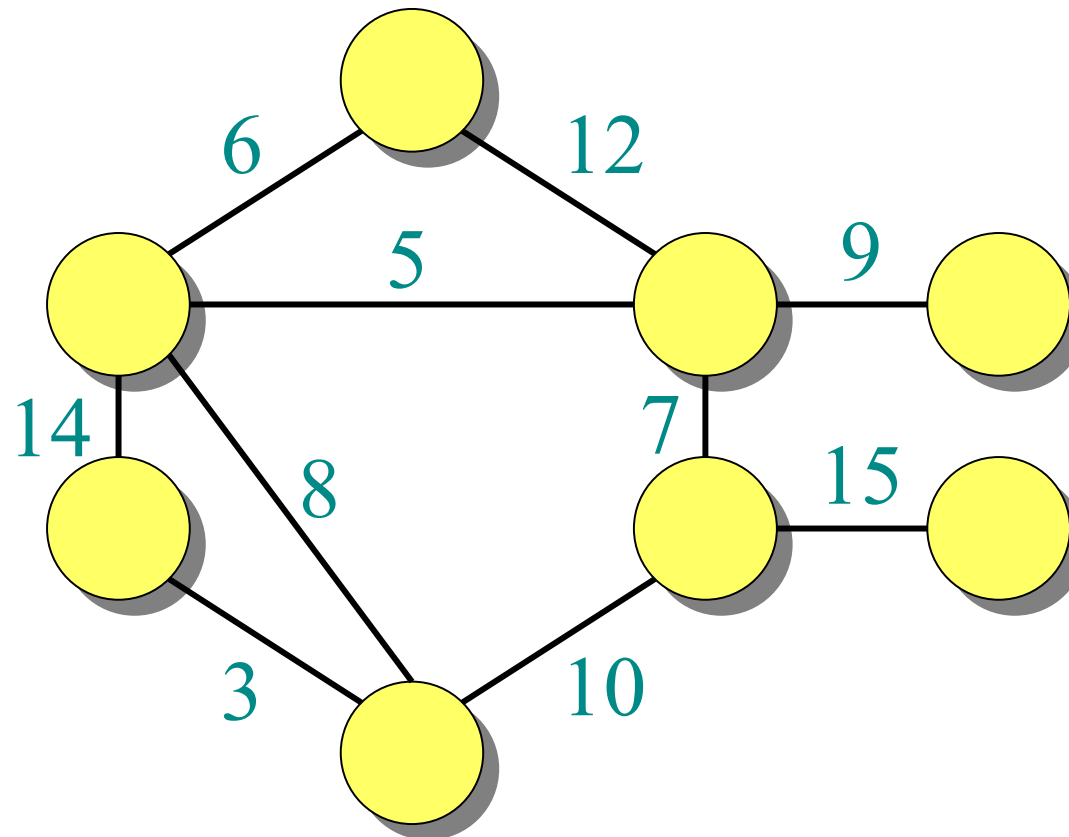
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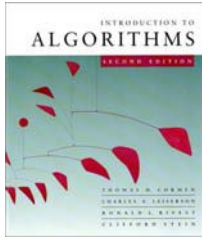
Output: A *spanning tree* T — a tree that connects all vertices — of minimum weight:

$$w(T) = \sum_{(u,v) \in T} w(u,v).$$

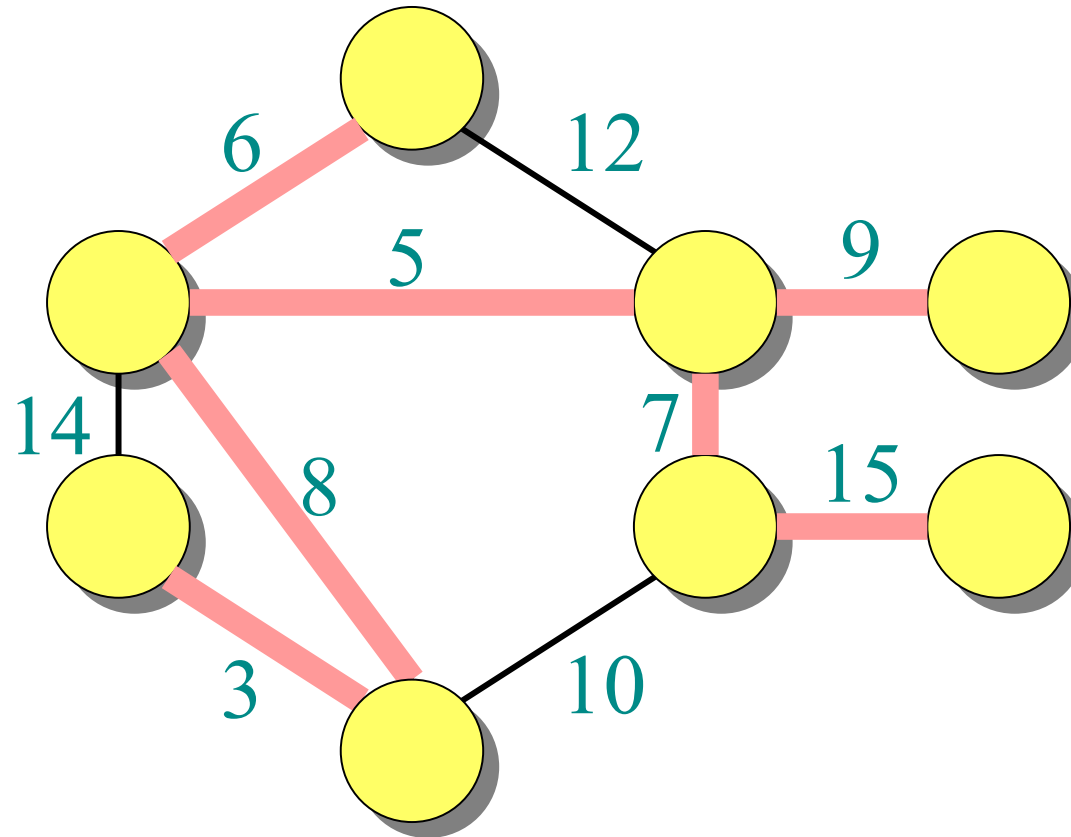


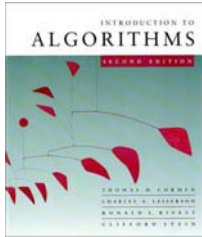
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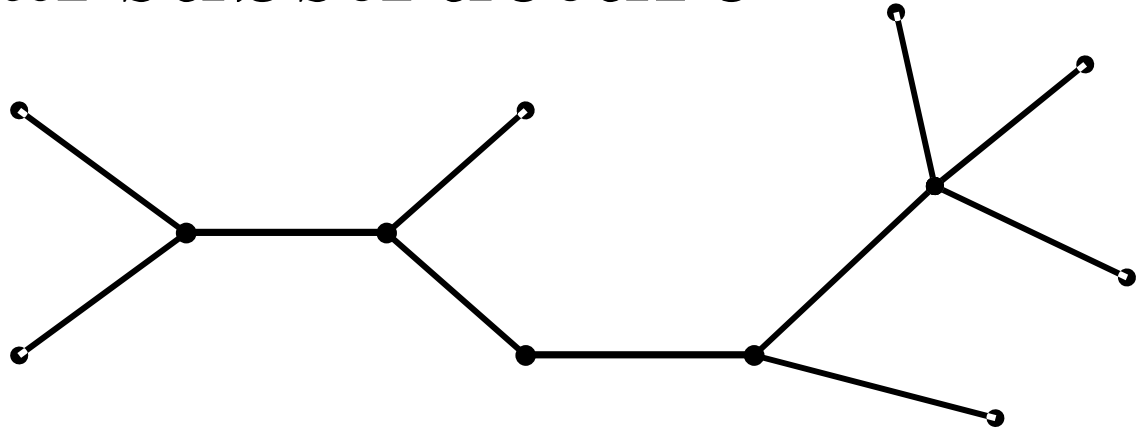


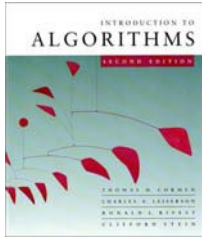


Optimal substructure

MST T :

(Other edges of G
are not shown.)

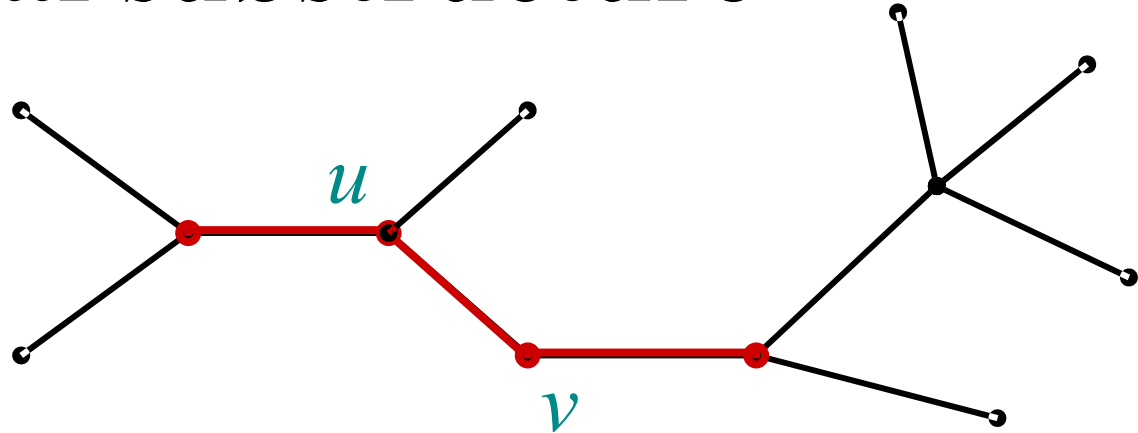




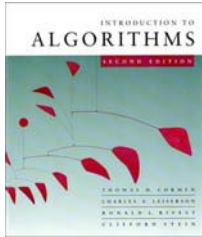
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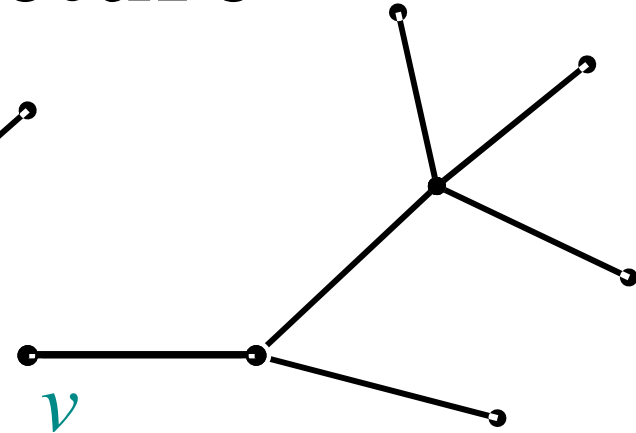
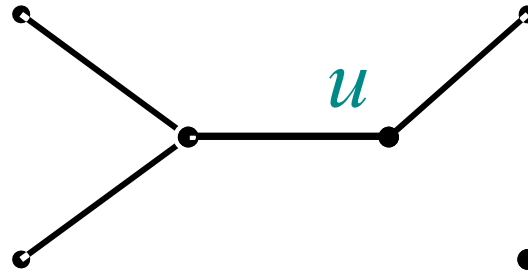
Remove any edge $(u, v) \in T$.



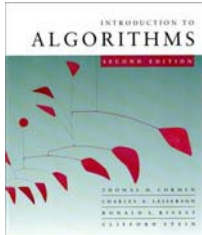
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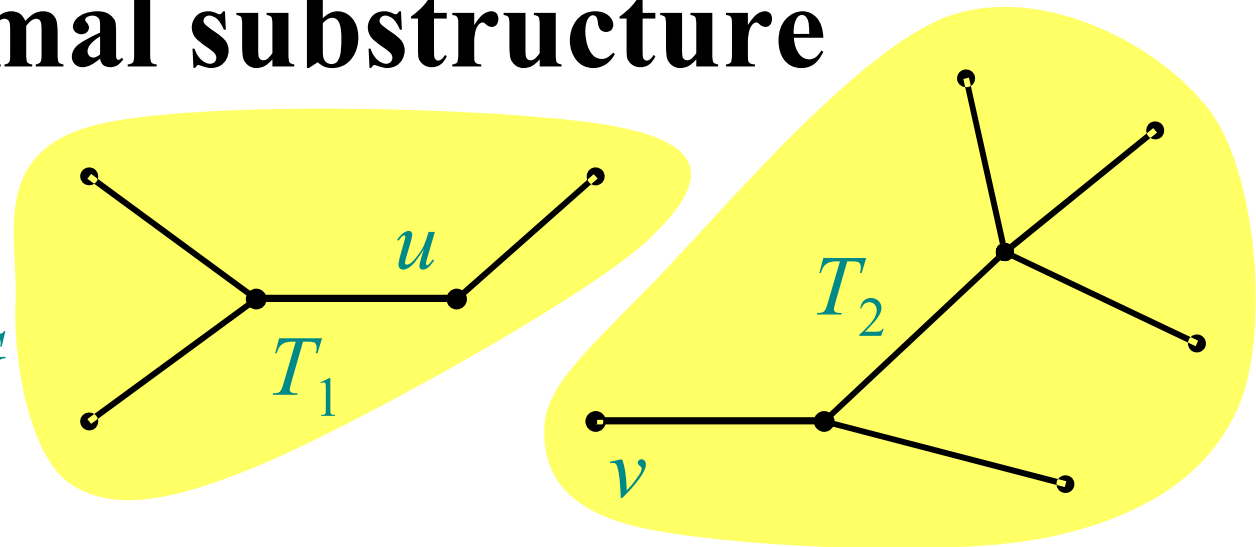
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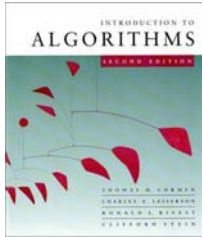
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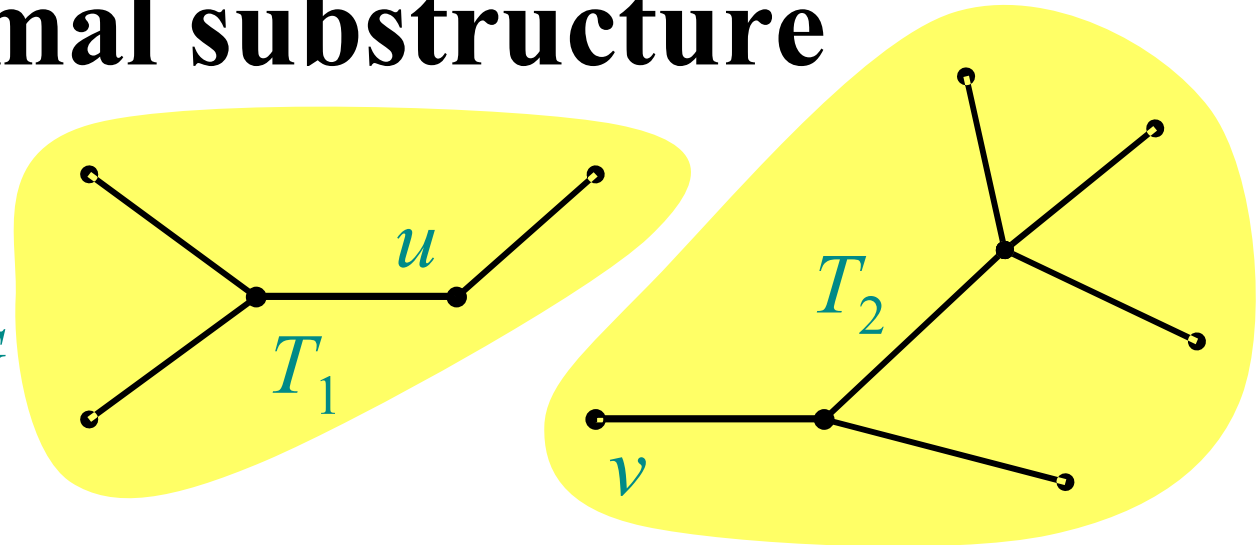
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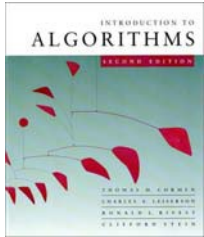


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Theorem. The subtree T_1 is an MST of $G_1 = (V_1, E_1)$, the subgraph of G *induced* by the vertices of T_1 :

$$V_1 = \text{vertices of } T_1,$$
$$E_1 = \{ (x, y) \in E : x, y \in V_1 \}.$$

Similarly for T_2 .

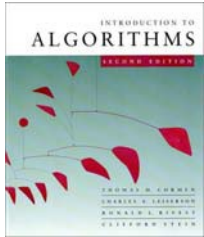


Proof of optimal substructure

Proof. Cut and paste:

$$w(T) = w(u, v) + w(T_1) + w(T_2).$$

If T_1' were a lower-weight spanning tree than T_1 for G_1 , then $T' = \{(u, v)\} \cup T_1' \cup T_2$ would be a lower-weight spanning tree than T for G . □



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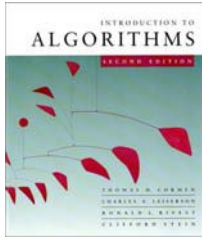
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Do we also have overlapping subproblems?

- Yes.



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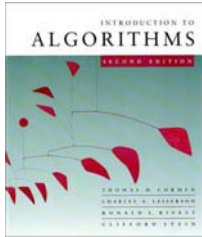
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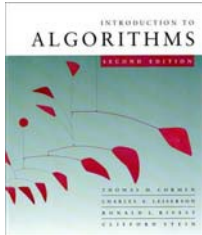
Great, then dynamic programming may work!

- Yes, but MST exhibits another powerful property which leads to an even more efficient algorithm.



Hallmark for “greedy” algorithms

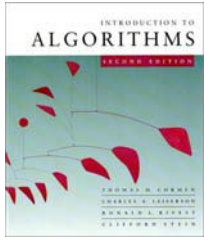
Greedy-choice property
*A locally optimal choice
is globally optimal.*



Hallmark for “greedy” algorithms

***Greedy-choice property**
A locally optimal choice
is globally optimal.*

Theorem. Let T be the MST of $G = (V, E)$, and let $A \subseteq V$. Suppose that $(u, v) \in E$ is the least-weight edge connecting A to $V - A$. Then, $(u, v) \in T$.

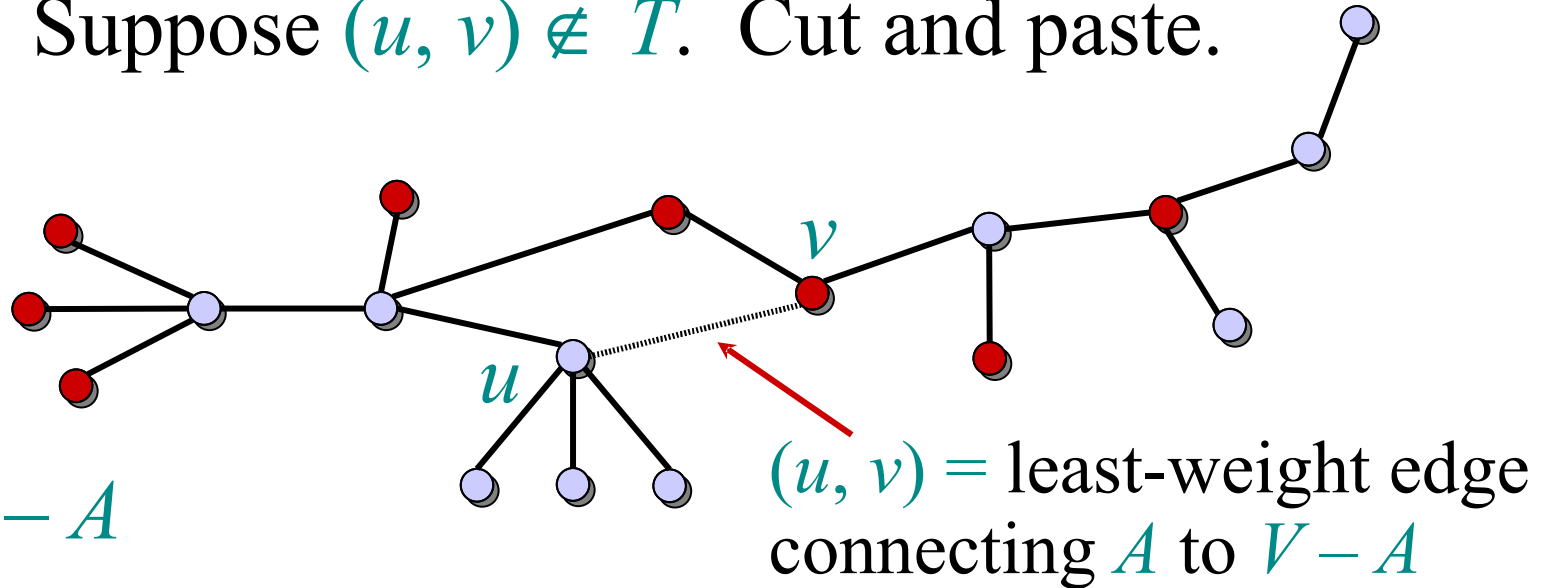


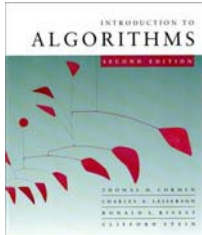
Proof of theorem

Proof. Suppose $(u, v) \notin T$. Cut and paste.

T :

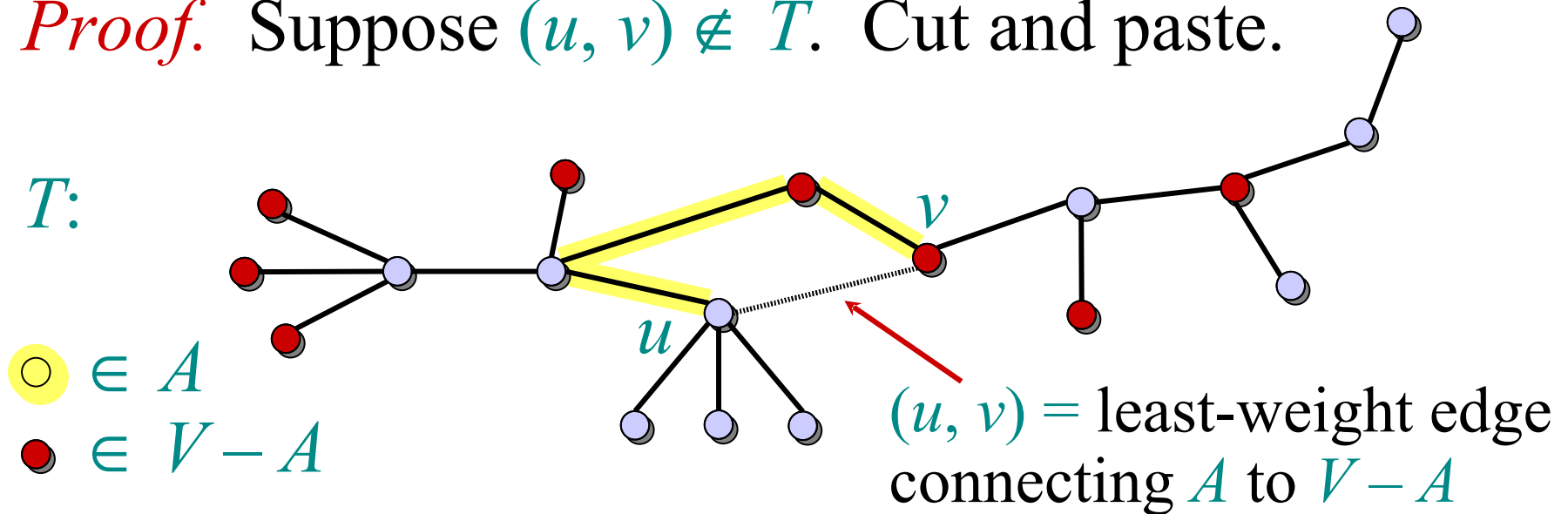
$\bullet \in A$
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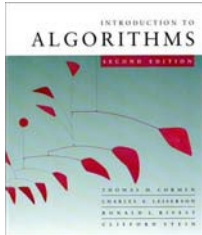


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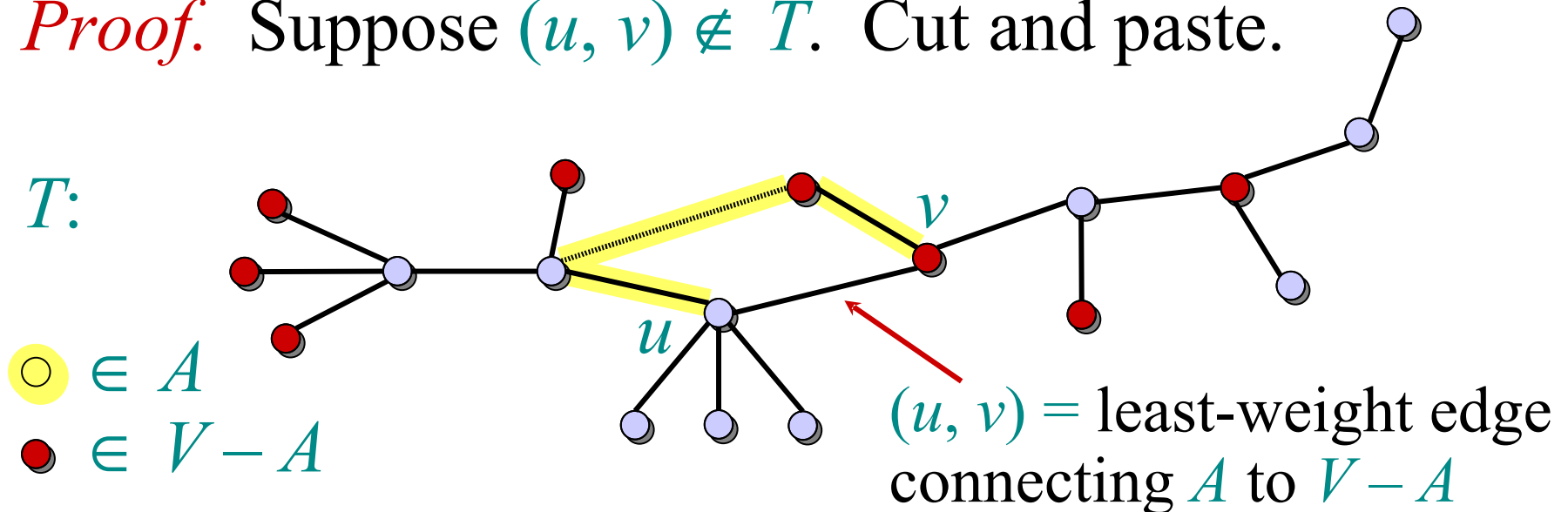


Consider the unique simple path from u to v in T .



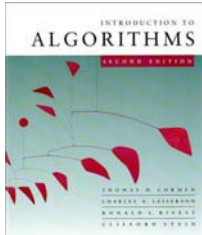
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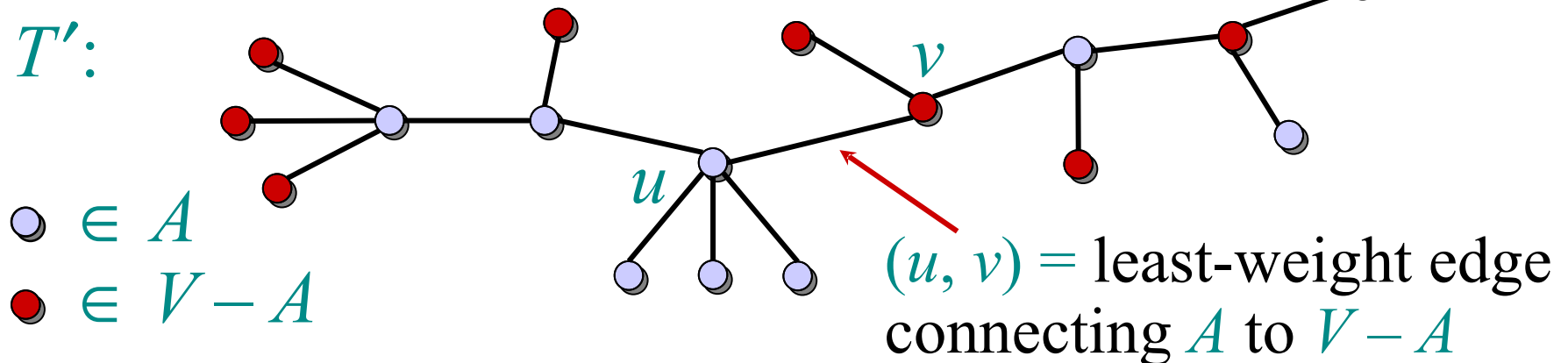
Swap (u, v) with the first edge on this path that connects a vertex in A to a vertex in $V - A$.



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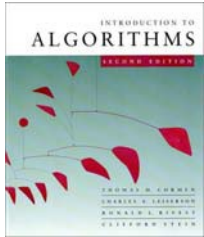
T' :



Consider the unique simple path from u to v in T .

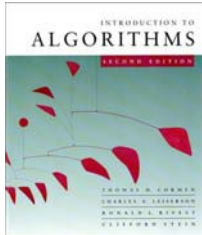
Swap (u, v) with the first edge on this path that connects a vertex in A to a vertex in $V - A$.

A lighter-weight spanning tree than T results. □



Prim's algorithm

IDEA: Maintain $V - A$ as a priority queue Q . Key each vertex in Q with the weight of the least-weight edge connecting it to a vertex in A .



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$Q \leftarrow V$

$key[v] \leftarrow \infty$ for all $v \in V$

$key[s] \leftarrow 0$ for some arbitrary $s \in V$

while $Q \neq \emptyset$

do $u \leftarrow \text{EXTRACT-MIN}(Q)$

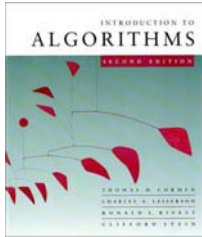
for each $v \in \text{Adj}[u]$

do if $v \in Q$ and $w(u, v) < key[v]$

then $key[v] \leftarrow w(u, v)$ ▷ DECREASE-KEY

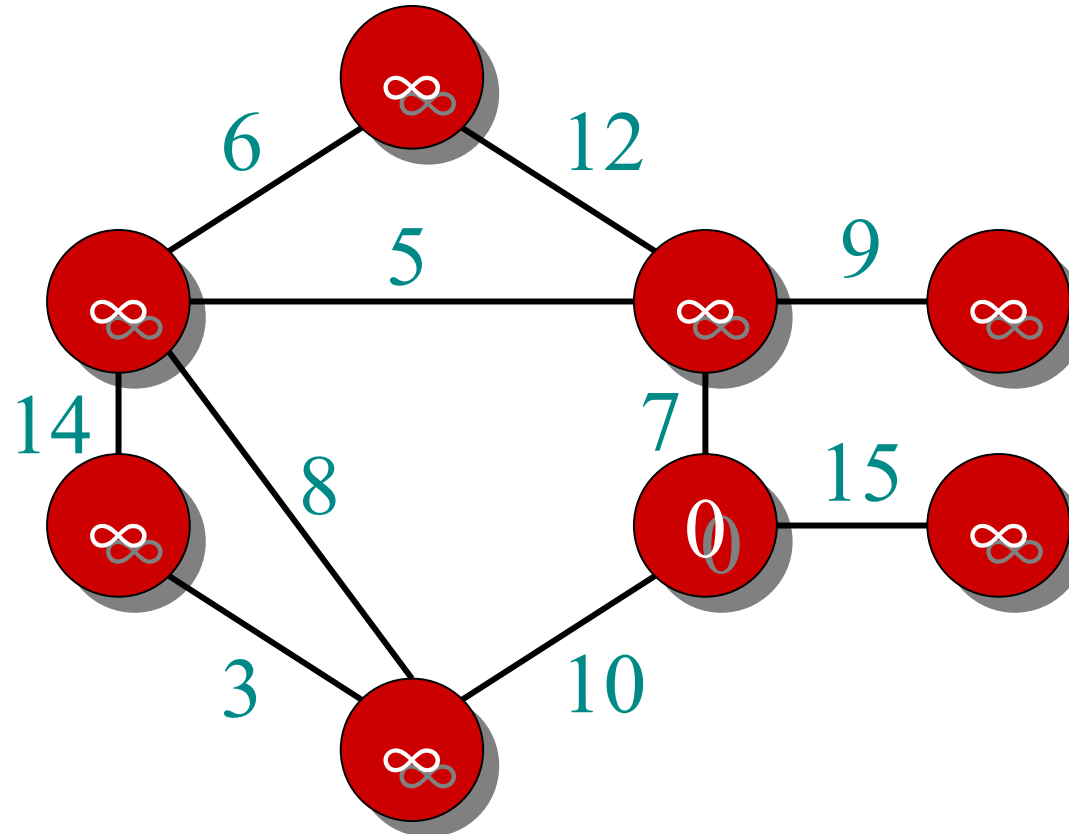
$\pi[v] \leftarrow u$

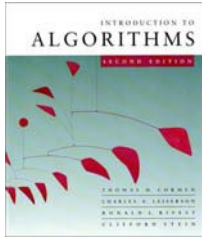
At the end, $\{(v, \pi[v])\}$ forms the MST.



Example of Prim's algorithm

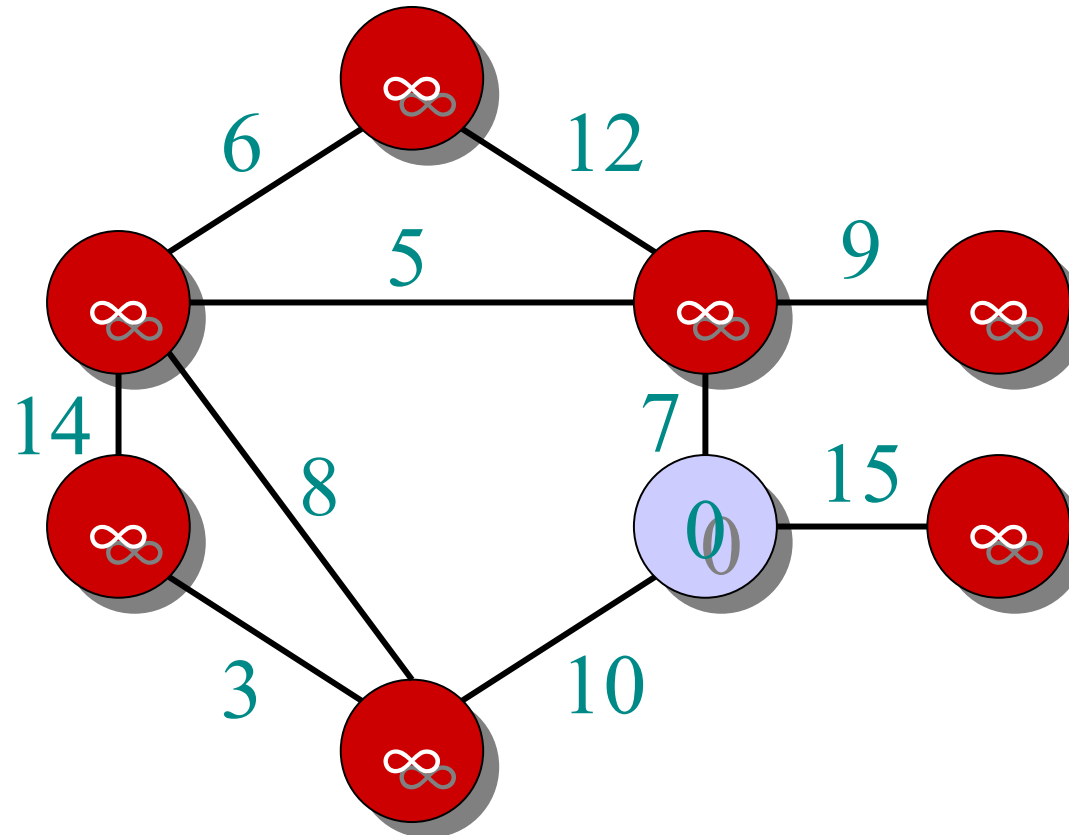
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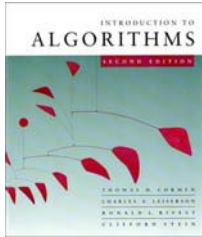




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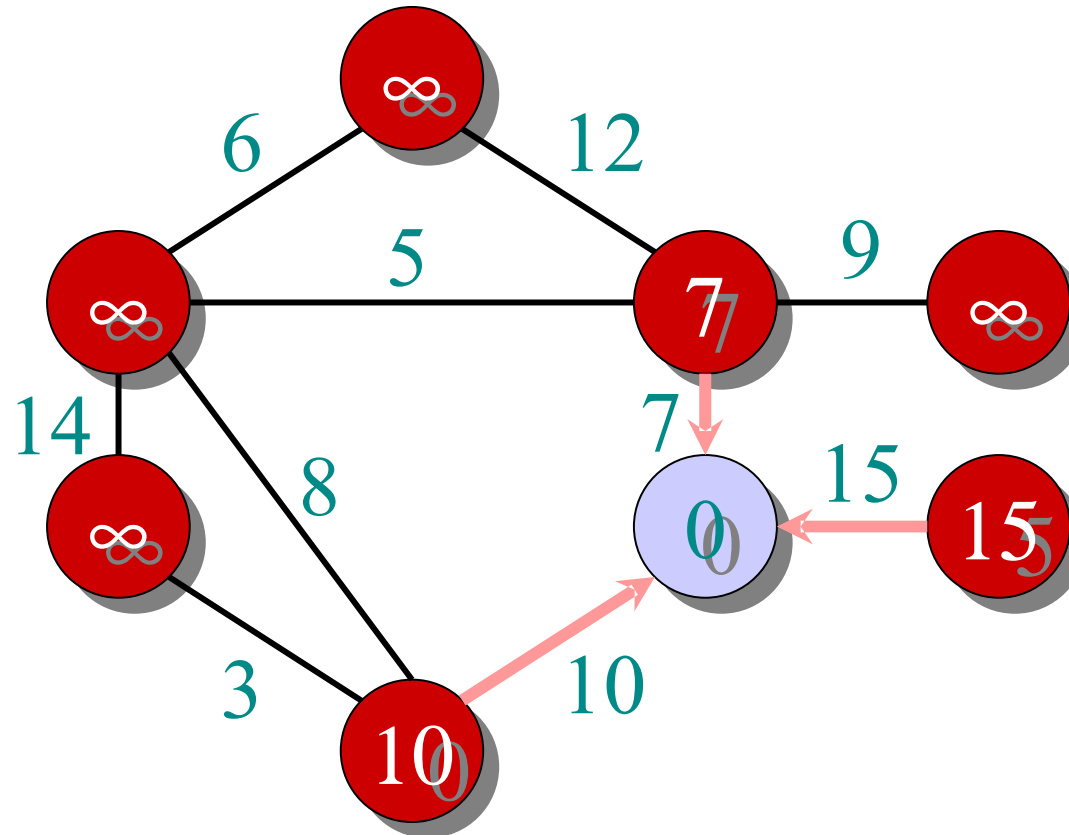
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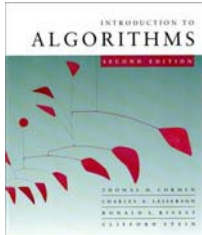




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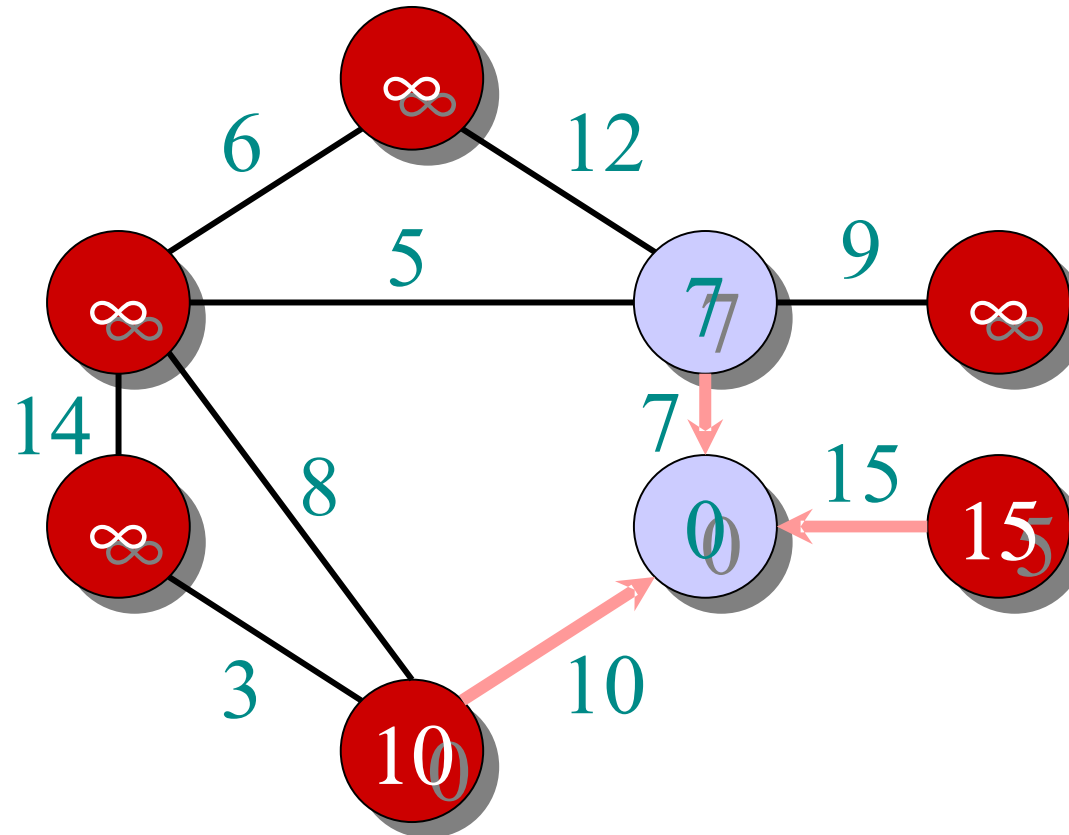
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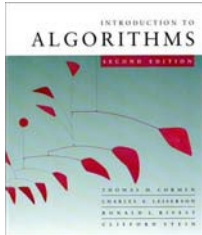




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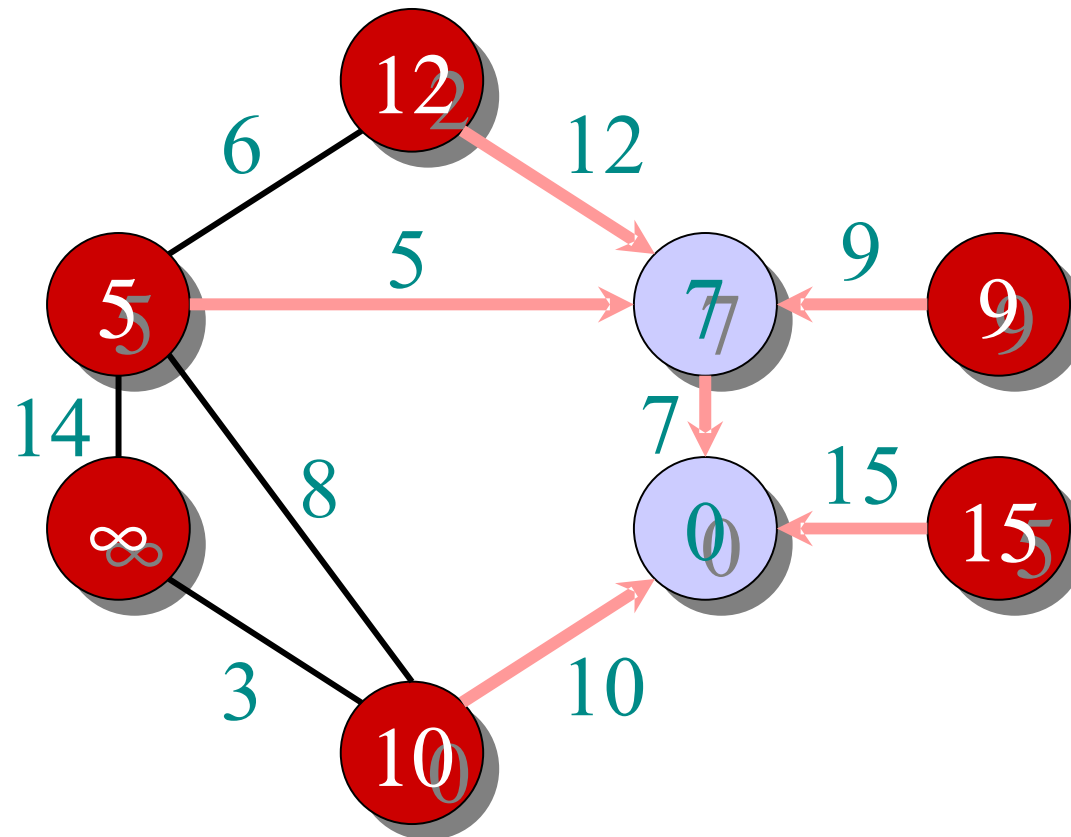
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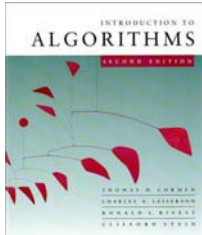




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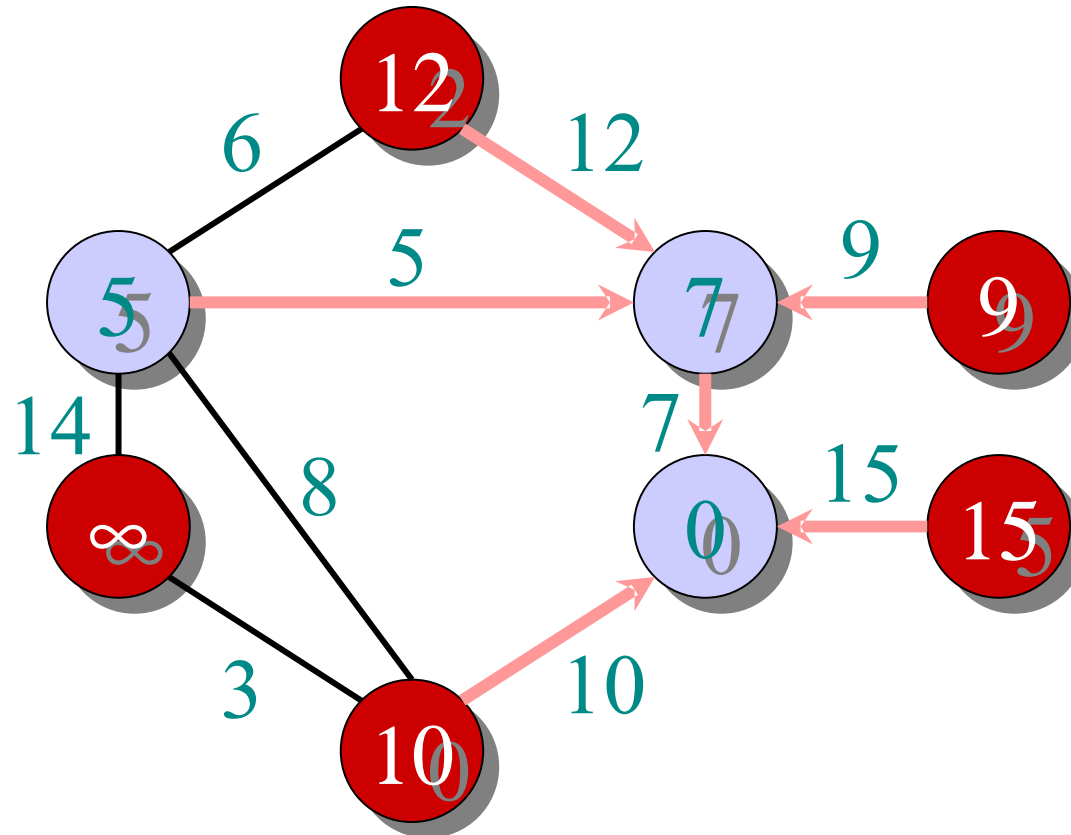
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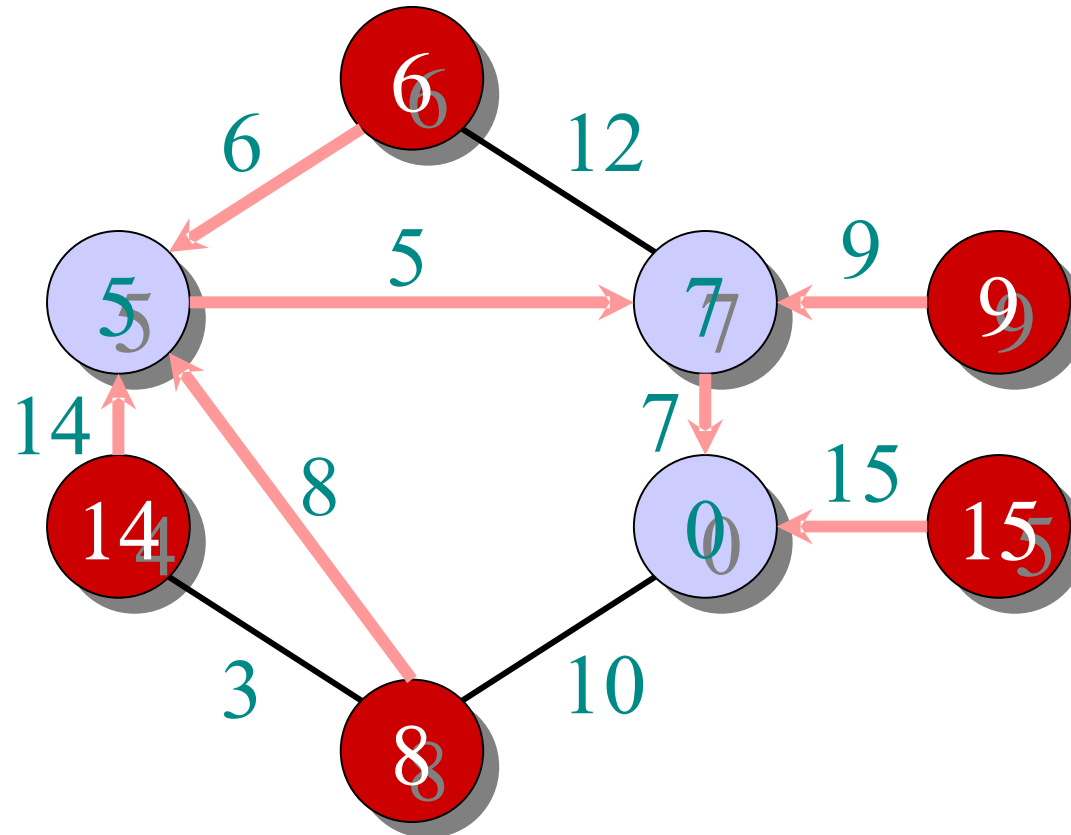
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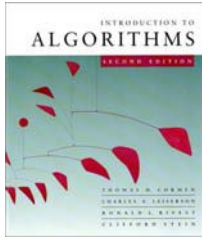




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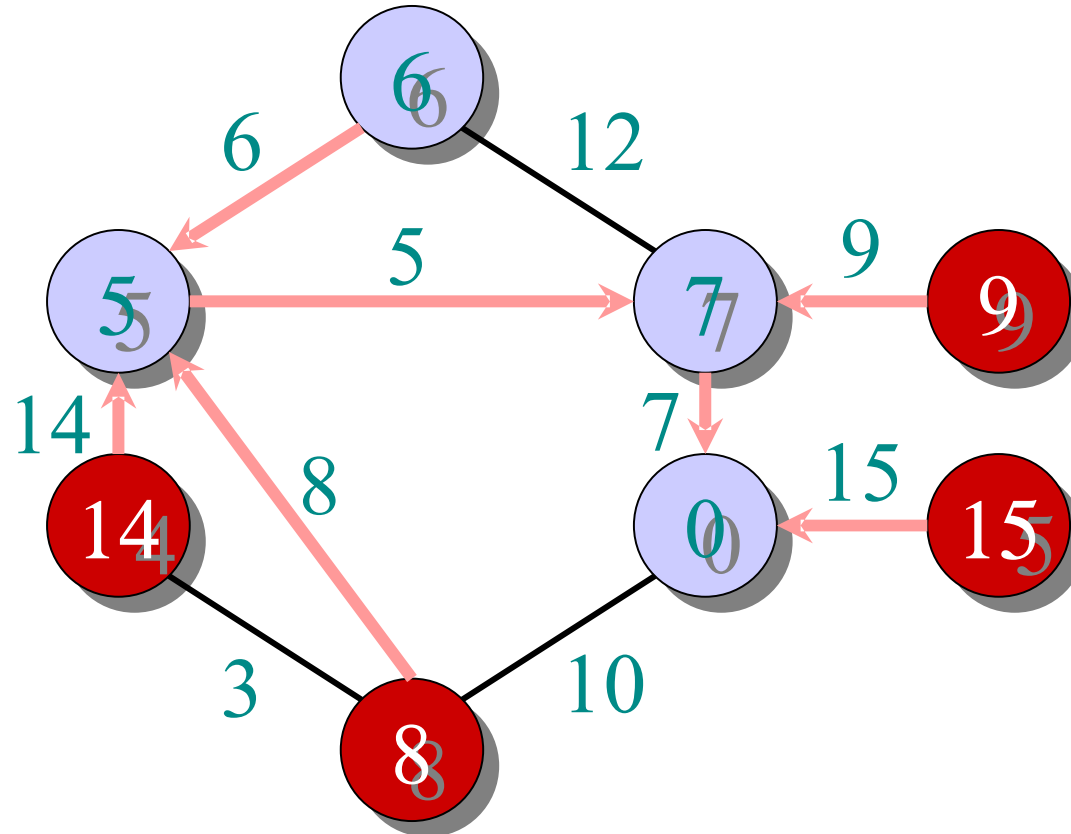
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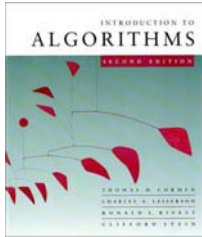




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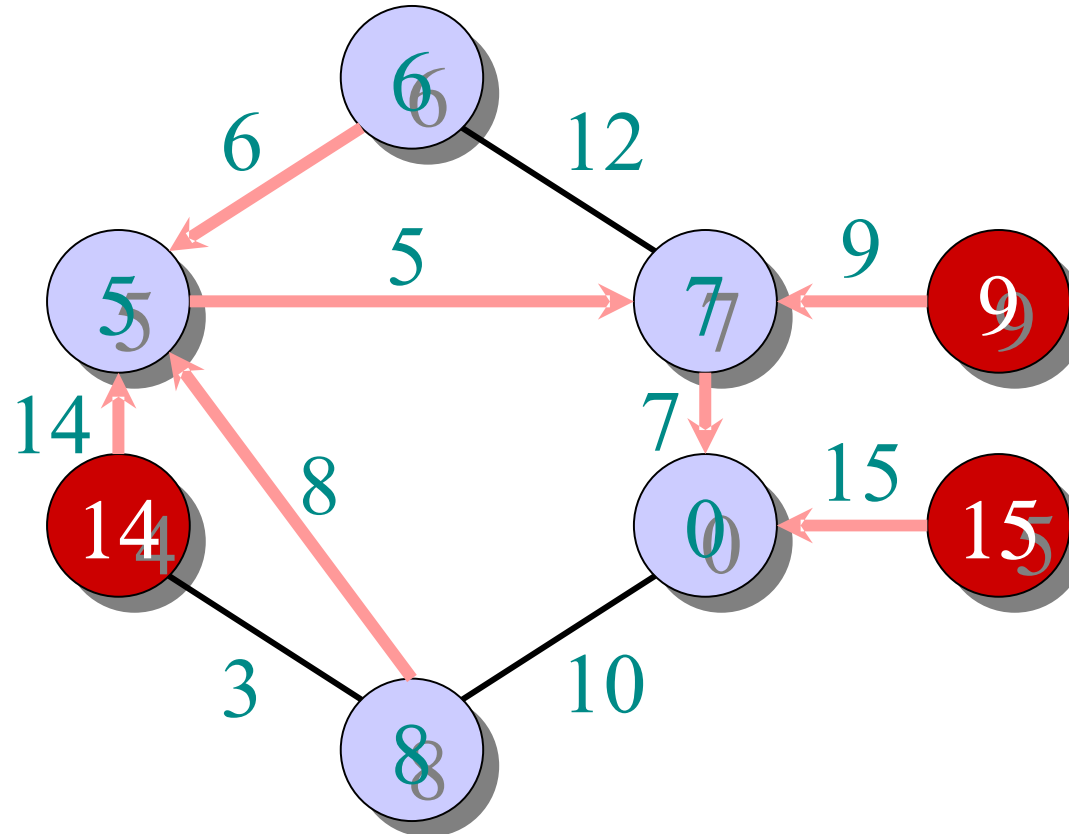
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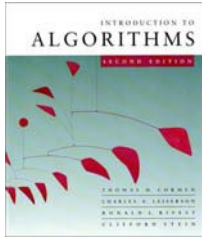




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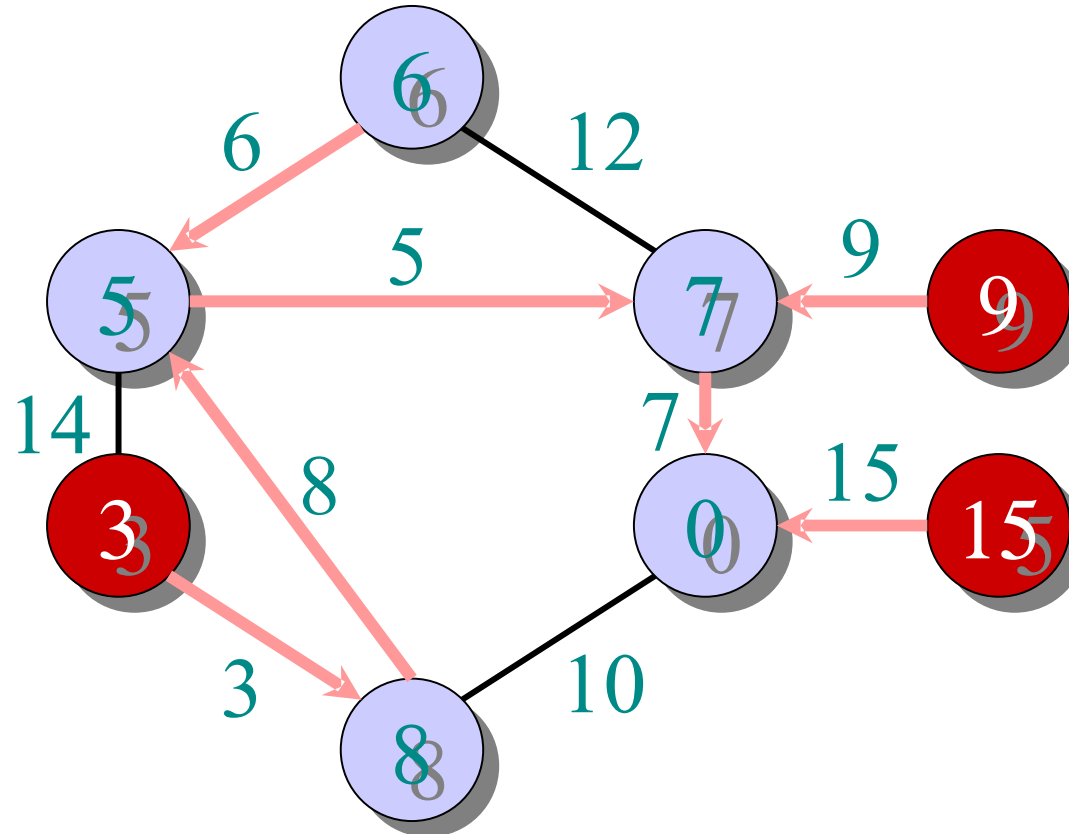
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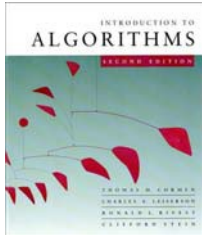




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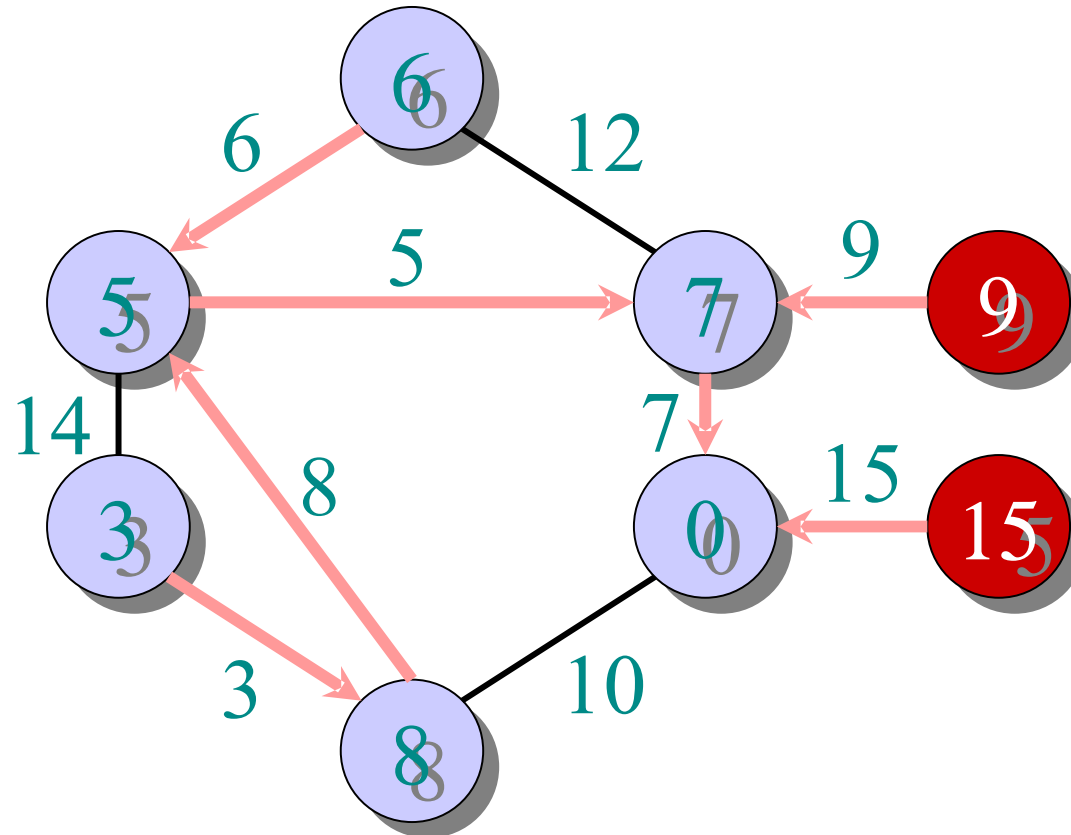
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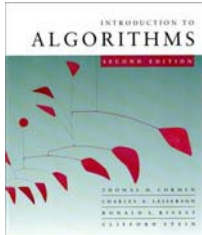




Example of Prim's algorithm

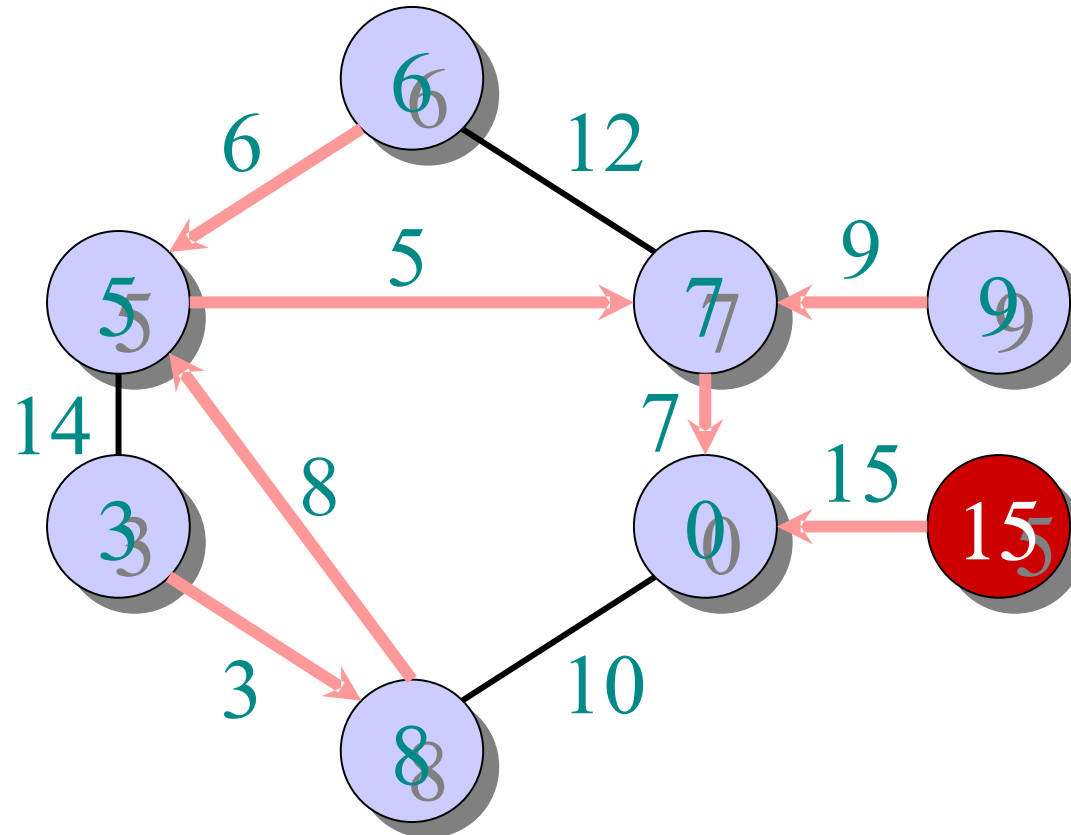
-  $\in A$
-  $\in V - A$

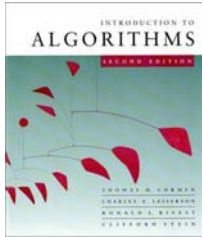




Example of Prim's algorithm

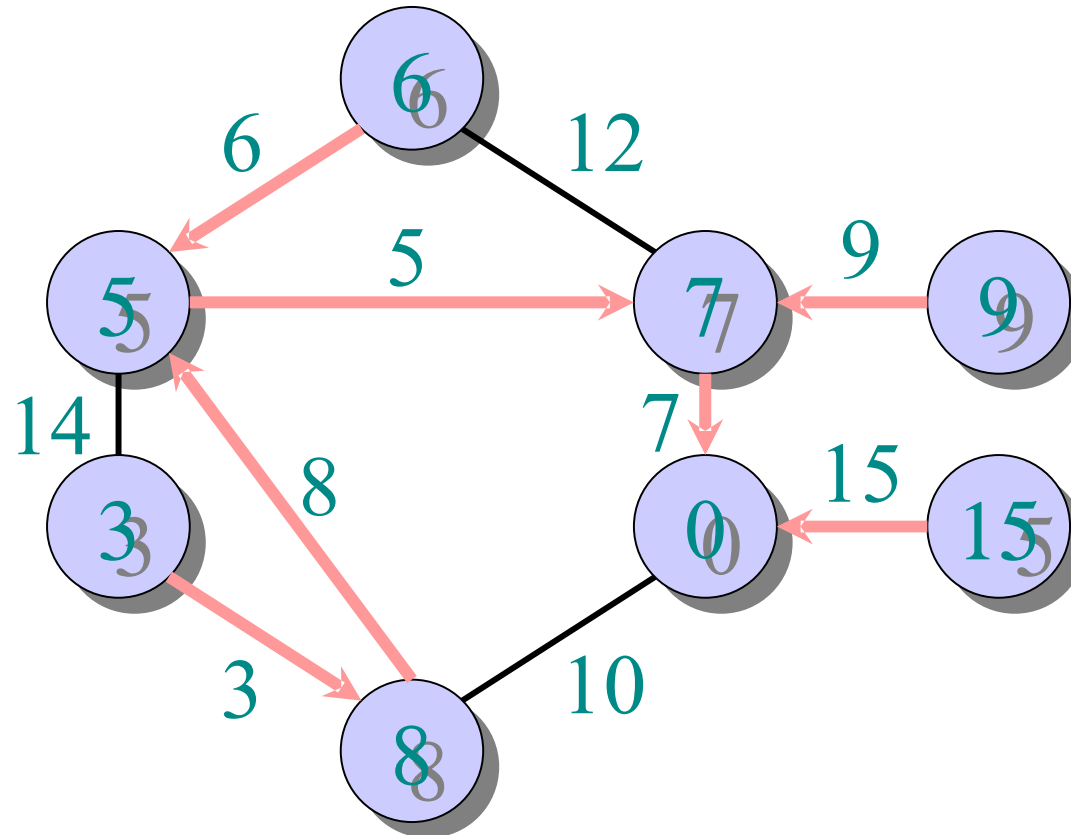
-  $\in A$
-  $\in V - A$

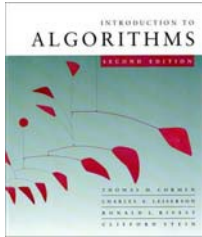




Example of Prim's algorithm

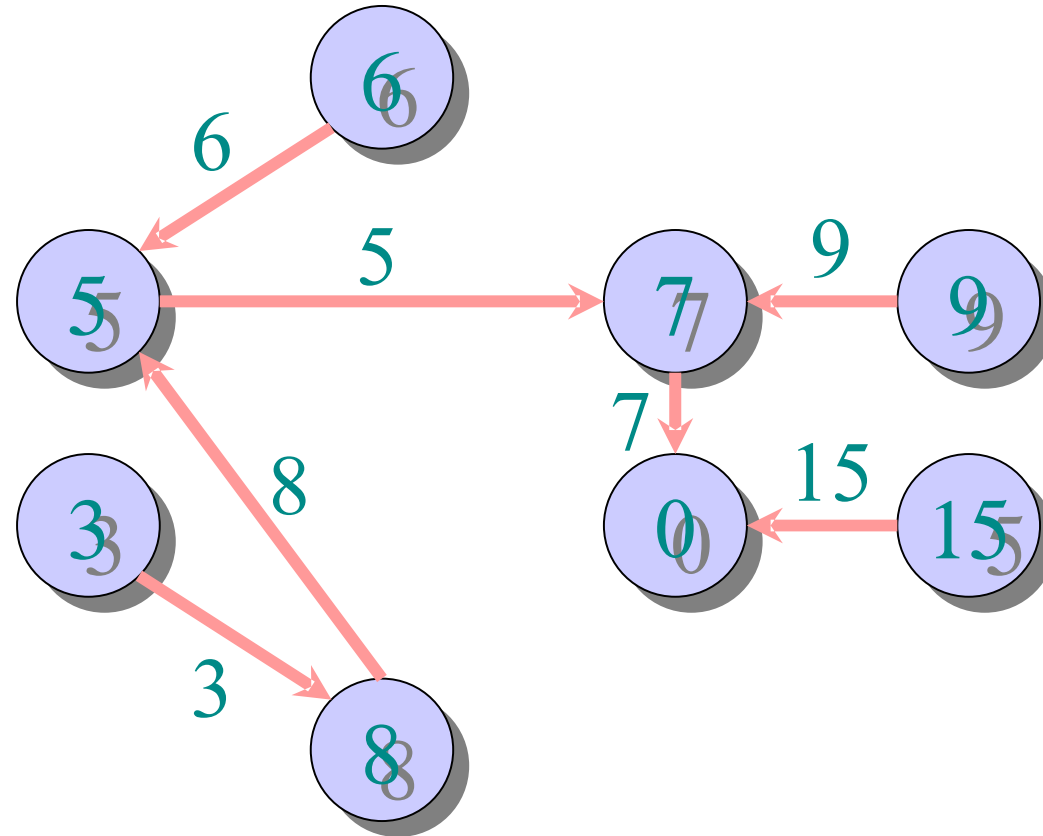
-  $\in A$
-  $\in V - A$





Example of Prim's algorithm

-  $\in A$
-  $\in V - A$





Analysis of Prim

$Q \leftarrow V$

$key[v] \leftarrow \infty$ for all $v \in V$

$key[s] \leftarrow 0$ for some arbitrary $s \in V$

while $Q \neq \emptyset$

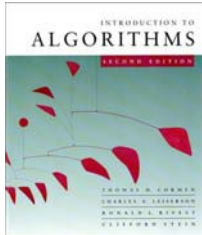
do $u \leftarrow \text{EXTRACT-MIN}(Q)$

for each $v \in \text{Adj}[u]$

do if $v \in Q$ and $w(u, v) < key[v]$

then $key[v] \leftarrow w(u, v)$

$\pi[v] \leftarrow u$



Analysis of Prim

$\Theta(V)$
total

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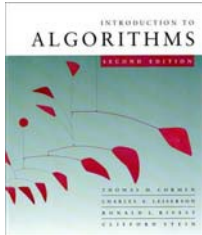


Analysis of Prim

$\Theta(V)$
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$|V|$
times

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while  $Q \neq \emptyset$   
  do  $u \leftarrow \text{EXTRACT-MIN}(Q)$   
    for each  $v \in \text{Adj}[u]$   
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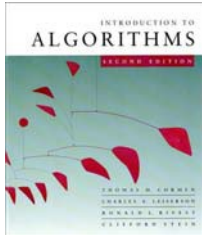
Analysis of Prim

$\Theta(V)$ total

$|V|$ times

$degree(u)$ times

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Analysis of Prim

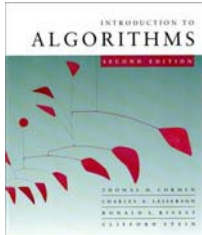
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```

Handshaking Lemma $\Rightarrow \Theta(E)$ implicit DECREASE-KEY's.



Analysis of Prim

$\Theta(V)$ total

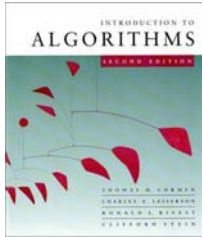
$|V|$ times

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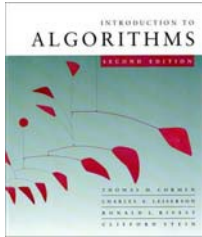
Handshaking Lemma $\Rightarrow \Theta(E)$ implicit DECREASE-KEY's.

$$\text{Time} = \Theta(V) \cdot T_{\text{EXTRACT-MIN}} + \Theta(E) \cdot T_{\text{DECREASE-KEY}}$$



Analysis of Prim (continued)

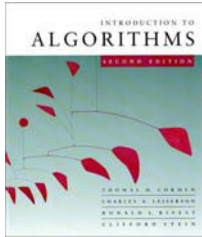
$$\text{Time} = \Theta(V) \cdot T_{\text{EXTRACT-MIN}} + \Theta(E) \cdot T_{\text{DECREASE-KEY}}$$



Analysis of Prim (continued)

$$\text{Time} = \Theta(V) \cdot T_{\text{EXTRACT-MIN}} + \Theta(E) \cdot T_{\text{DECREASE-KEY}}$$

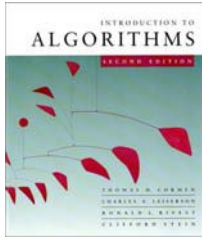
Q	$T_{\text{EXTRACT-MIN}}$	$T_{\text{DECREASE-KEY}}$	Total
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Analysis of Prim (continued)

$$\text{Time} = \Theta(V) \cdot T_{\text{EXTRACT-MIN}} + \Theta(E) \cdot T_{\text{DECREASE-KEY}}$$

Q	$T_{\text{EXTRACT-MIN}}$	$T_{\text{DECREASE-KEY}}$	Total
array	$O(V)$	$O(1)$	$O(V^2)$

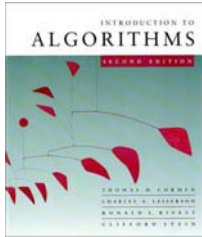


Analysis of Prim (continued)

$$\text{Time} = \Theta(V) \cdot T_{\text{EXTRACT-MIN}} + \Theta(E) \cdot T_{\text{DECREASE-KEY}}$$

Q	$T_{\text{EXTRACT-MIN}}$	$T_{\text{DECREASE-KEY}}$	Total
array	$O(V)$	$O(1)$	$O(V^2)$
binary heap	$O(\lg V)$	$O(\lg V)$	$O(E \lg V)$

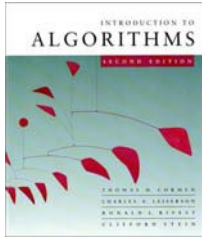
$O(V \lg V + E \lg V)$ and $E \geq V - 1$ (G is connected)



MST algorithms

Kruskal's algorithm (see CLRS):

- Uses the *disjoint-set data structure*
- Running time = $O(E \lg V)$.



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- Running time = $O(E \lg V)$.

Best to date:

- Karger, Klein, and Tarjan [1993].
- Randomized algorithm.
- $O(V + E)$ expected time.

Recap

- Greedy Algorithm Properties
- Graphs
 - Representation
 - Minimal Spanning Trees
- Next:
 - Amortized Analysis
 - Union-Find Data structure