CMP205: Computer Graphics

CAIRO UNIVERSITY

Lecture 14: Curves and Surfaces

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Fall 2012

Agenda

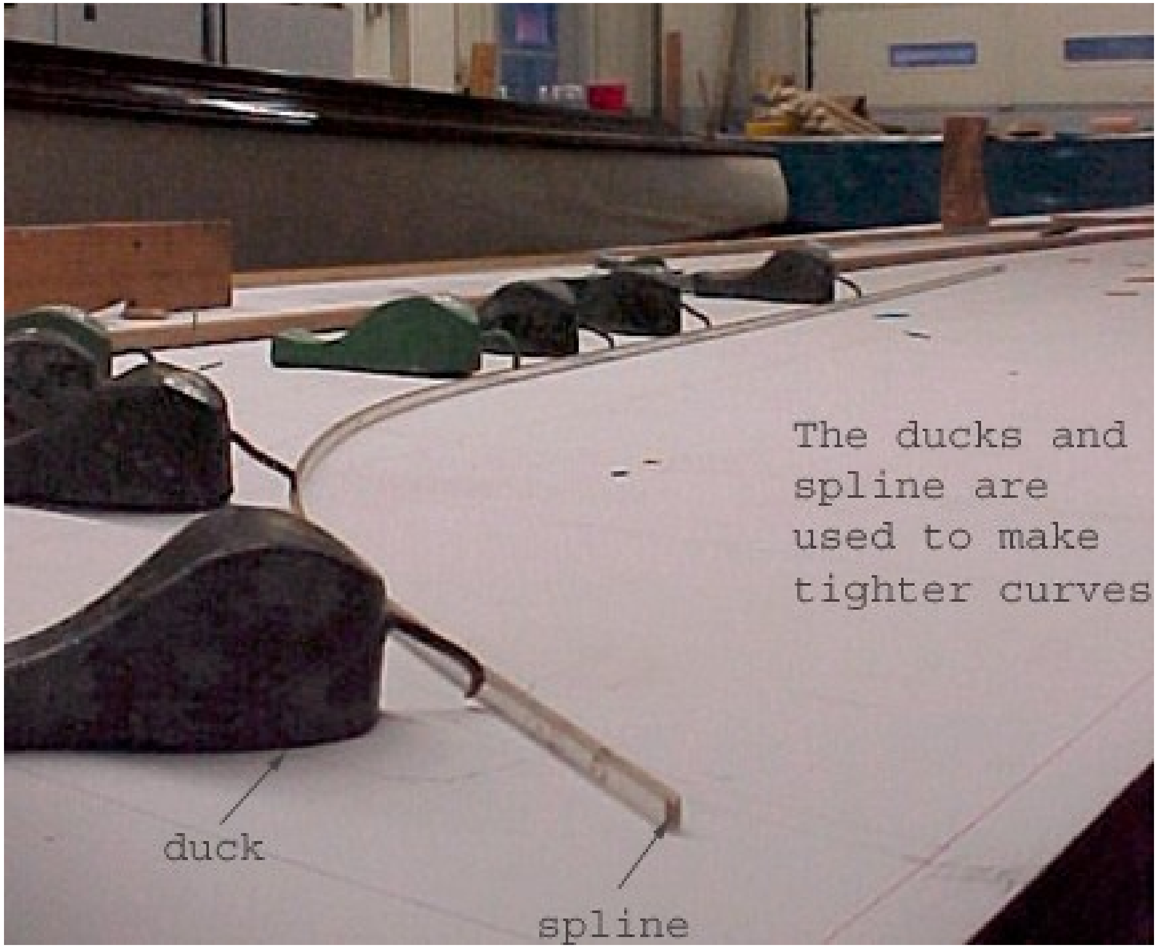
- Splines
  - Linear Splines
  - Hermite Splines
  - Bézier Splines
- Surfaces

Acknowledgment: Some slides adapted from Steve Marschner, Maneesh Agrawala, and Fredo Durand

Splines

- In many applications need to draw smooth curves
- So far
  - triangles, squares, …
  - circles, ellipses, ...

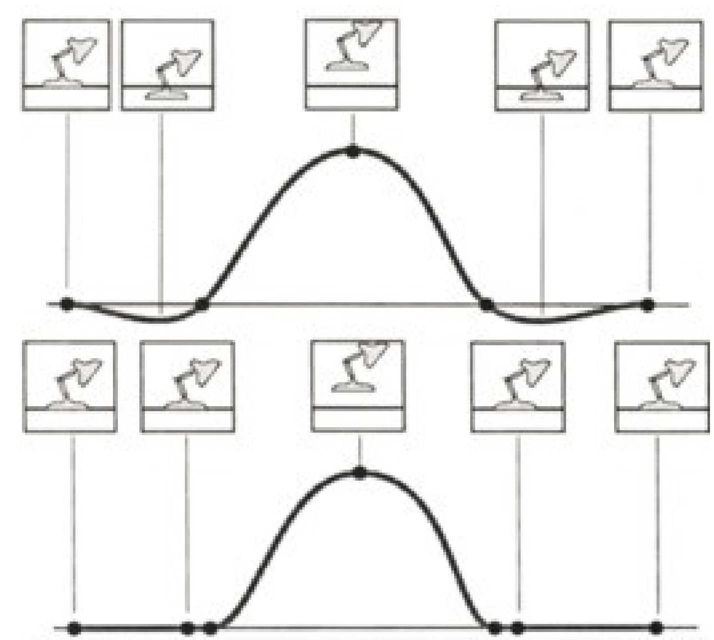
Splines



Metal Spline and Ducks used by draftsmen

Splines

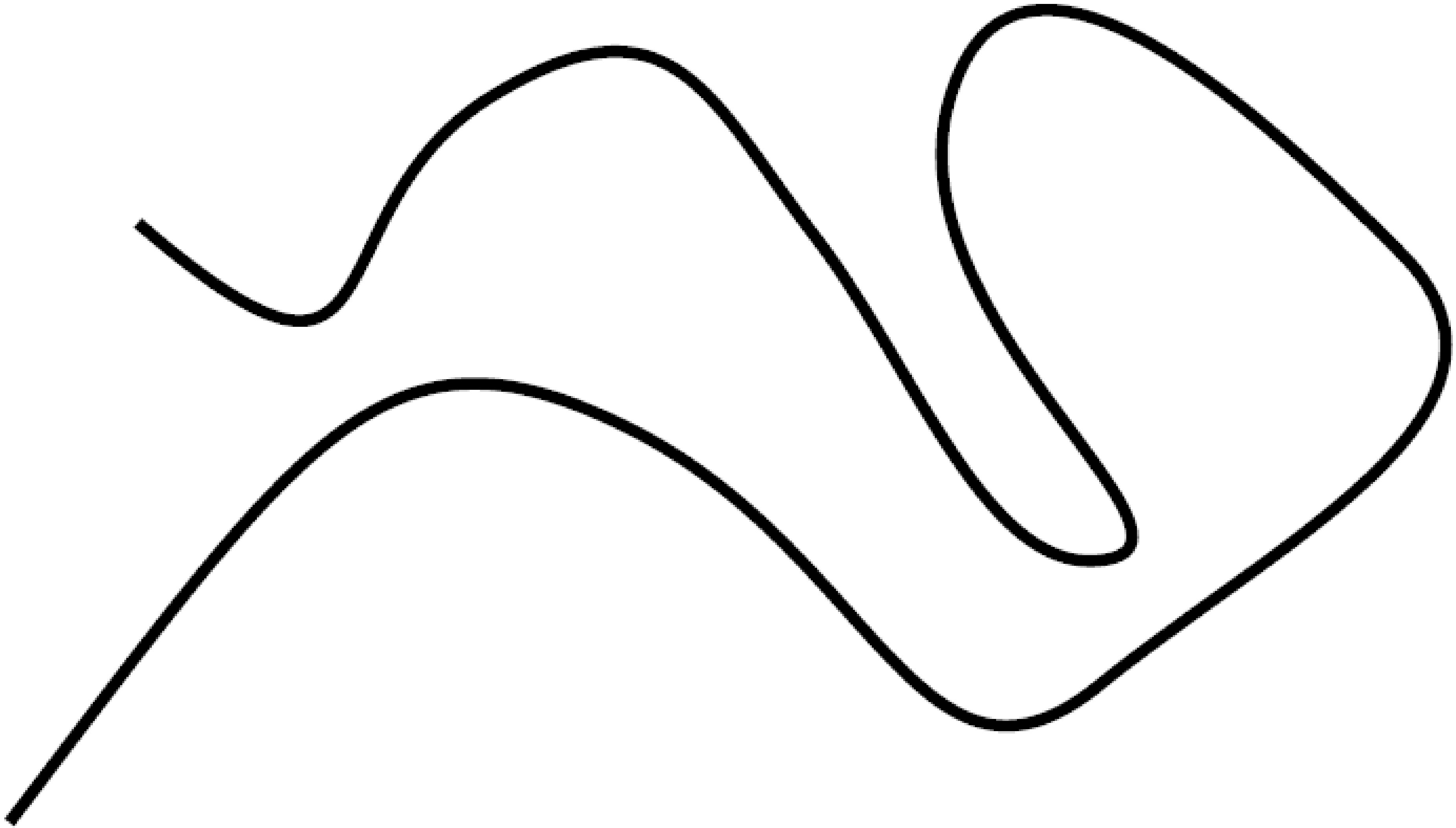
- Smooth curves
- Many applications
  - 2D modeling (Inkscape, Illustrator)
  - Fonts
  - 3D modeling
  - Animation
- Generally
  - Interpolation
  - Approximation



Splines

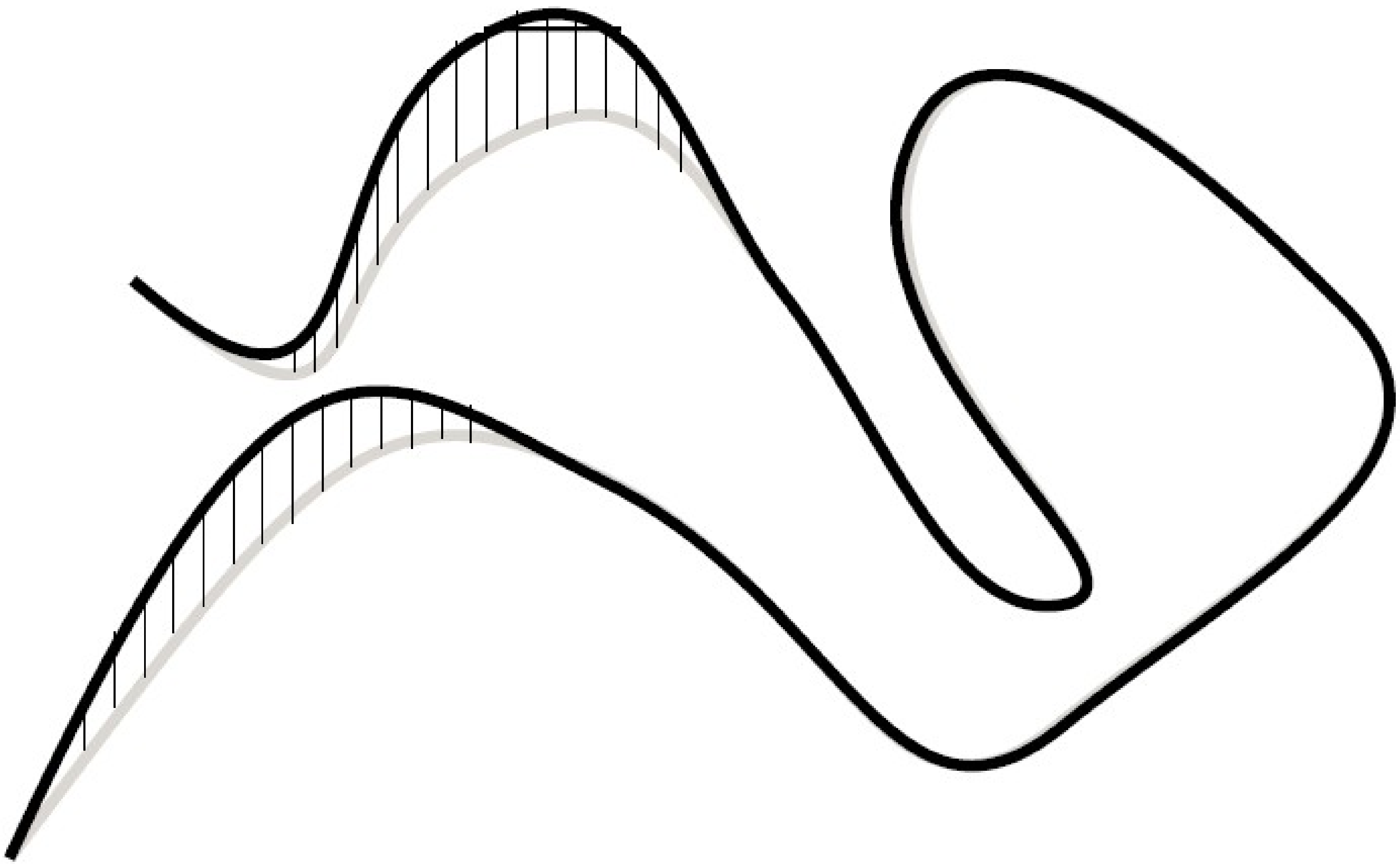
- Smoothness
  - Metal spline: metal curve minimization
  - Graphics: smooth functions (low order polynomial)
- Control
  - Metal spline: ducks
  - Graphics: control points

Splines



How many dimensions? 1D curve in 2D space

Splines

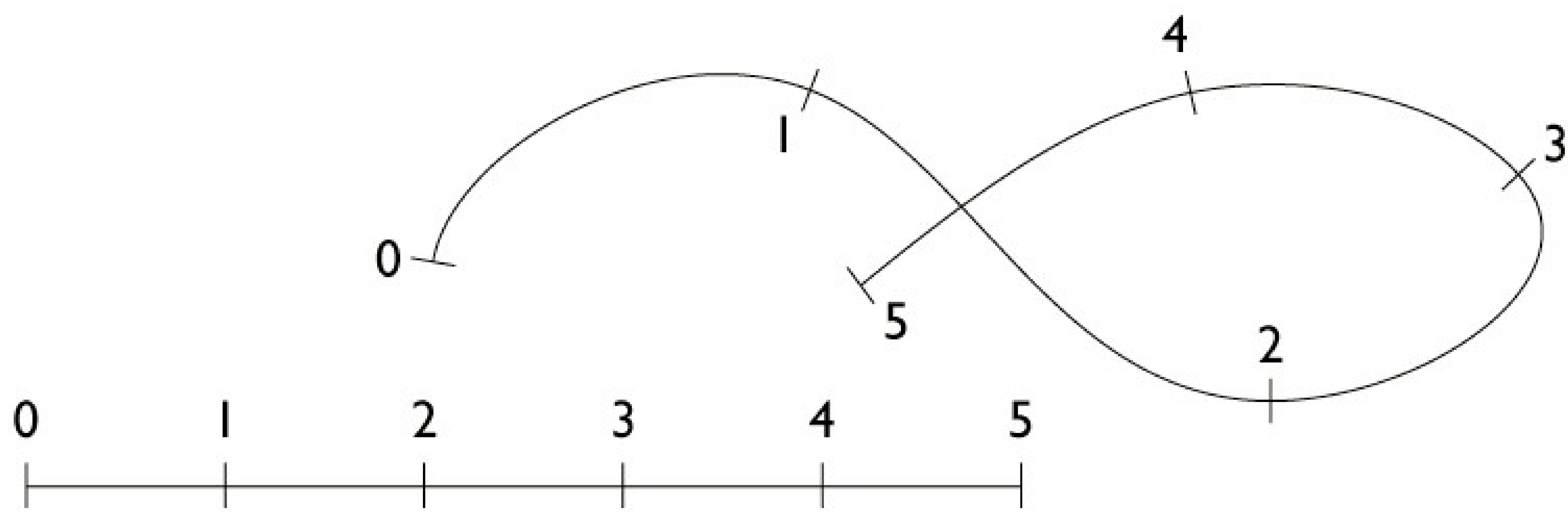


How many dimensions? 1D curve in 3D space

Splines

Parametric Curve $S = \{ \boldsymbol{p}(t) \mid t \in [0, N] \}$ $\boldsymbol{p}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$

Piecewise polynomial: different polynomial in each interval [i, i+1]

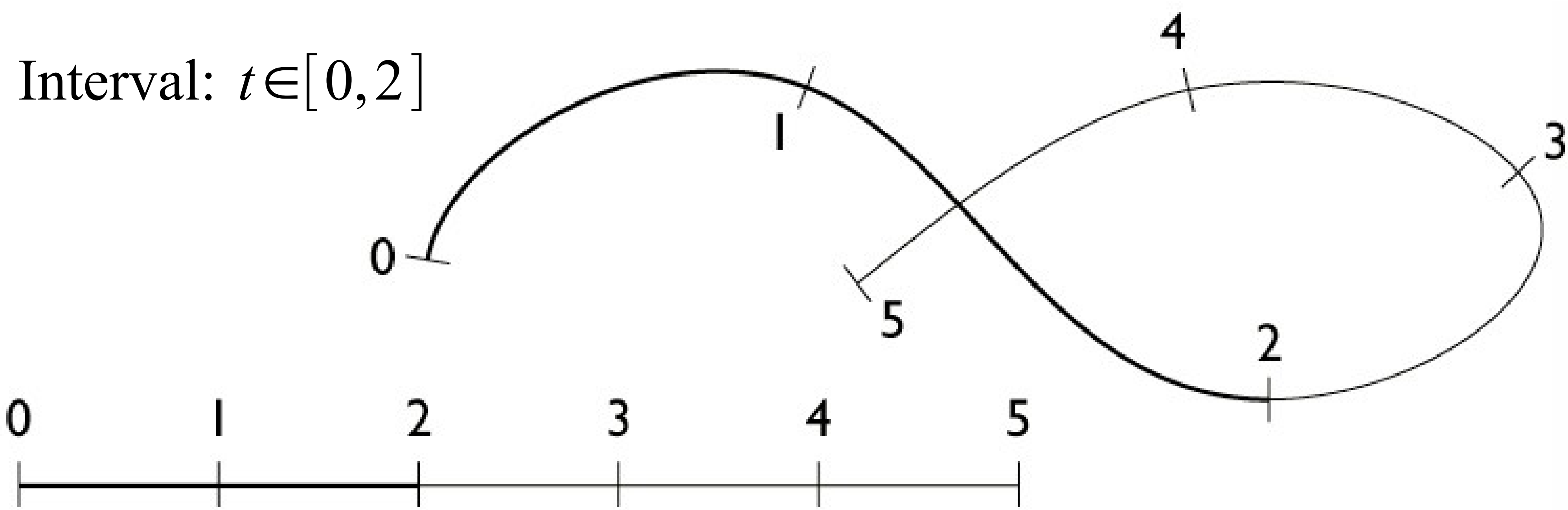


Splines

Parametric Curve $S = \{ \boldsymbol{p}(t) \mid t \in [0, N] \}$ $\boldsymbol{p}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$

Piecewise polynomial: different polynomial in each interval [i, i+1]

Interval: $t \in [0, 2]$



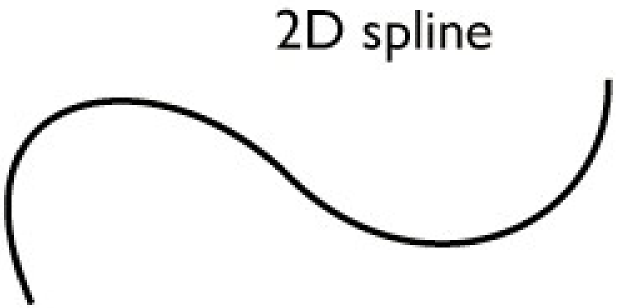
Splines

- Generally *f(t)* is piecewise polynomial
  - For example, cubic spline has cubic polynomials

$$x(t) = a_x t^3 + b_x t^2 + c_x t + d_x$$
$$y(t) = a_y t^3 + b_y t^2 + c_y t + d_y$$

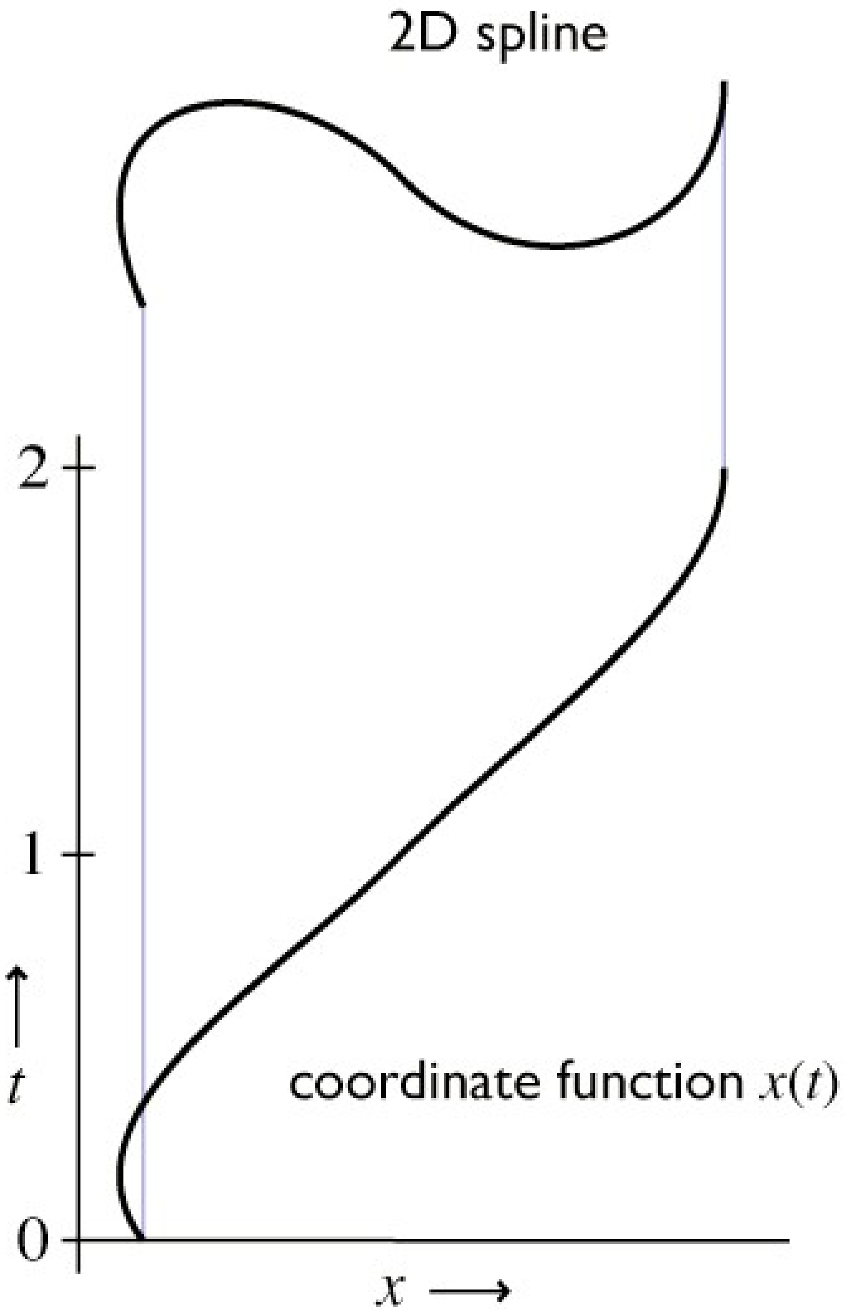
  - Coefficients different for every type of spline

Coordinate Functions



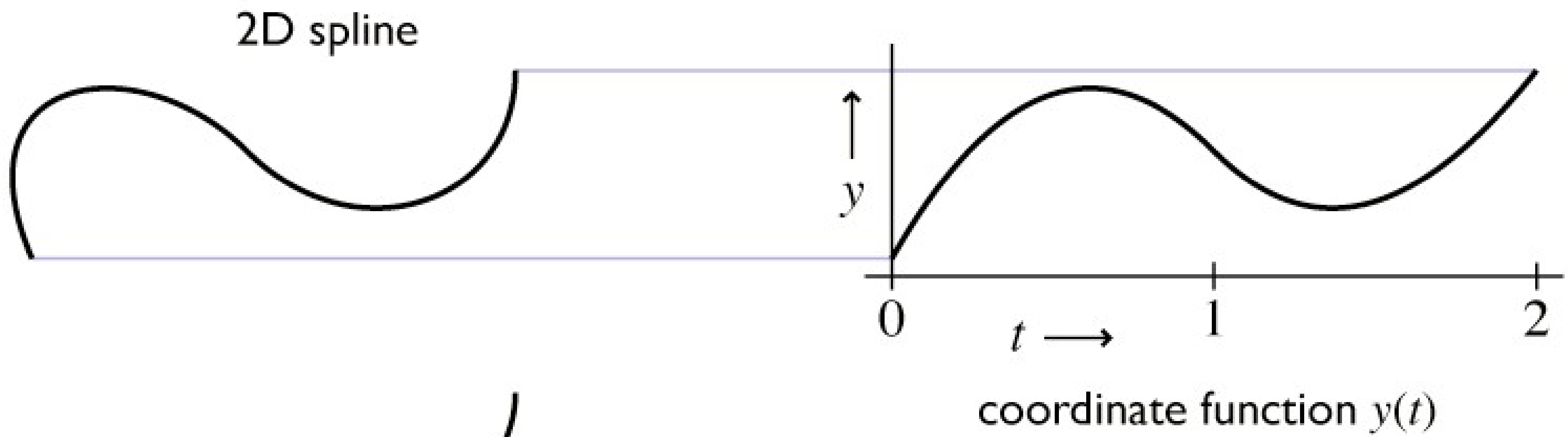
$$\boldsymbol{p}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

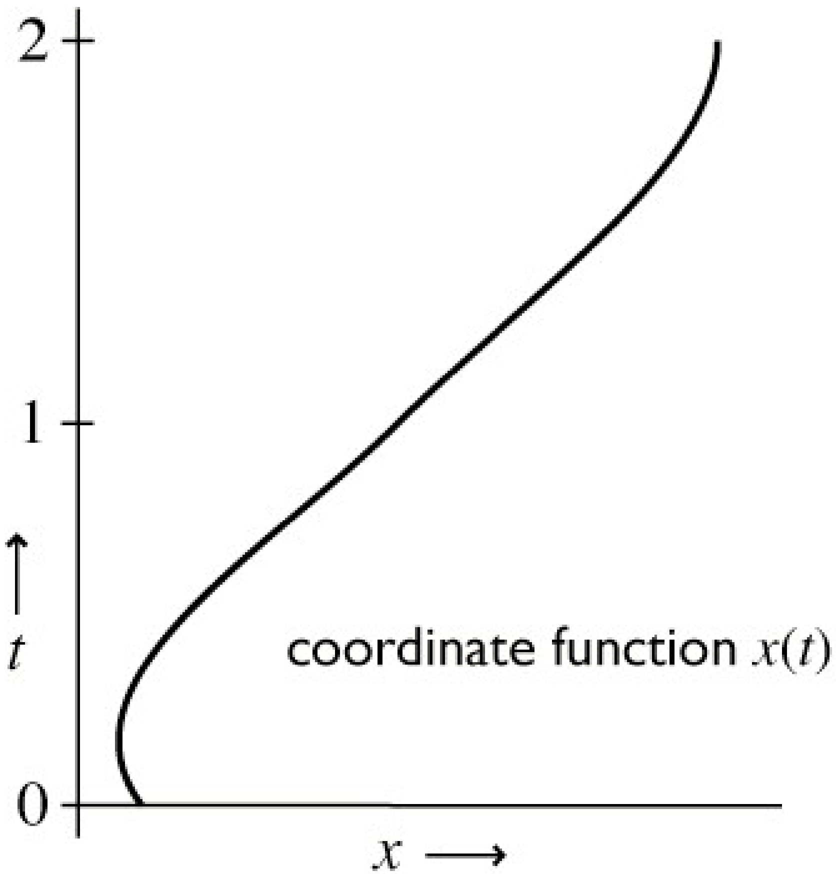
Coordinate Functions



$$\boldsymbol{p}(t)=\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

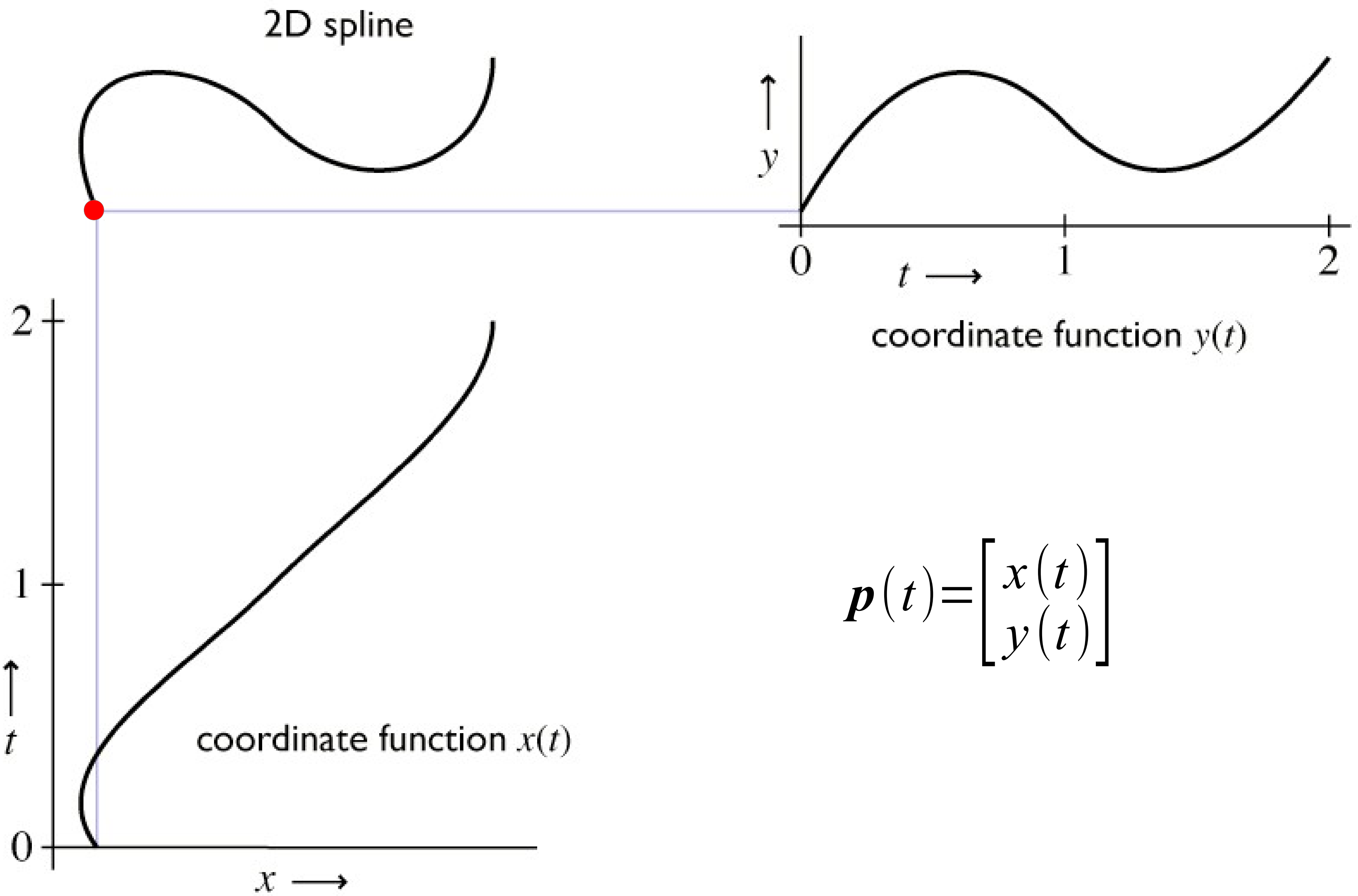
Coordinate Functions





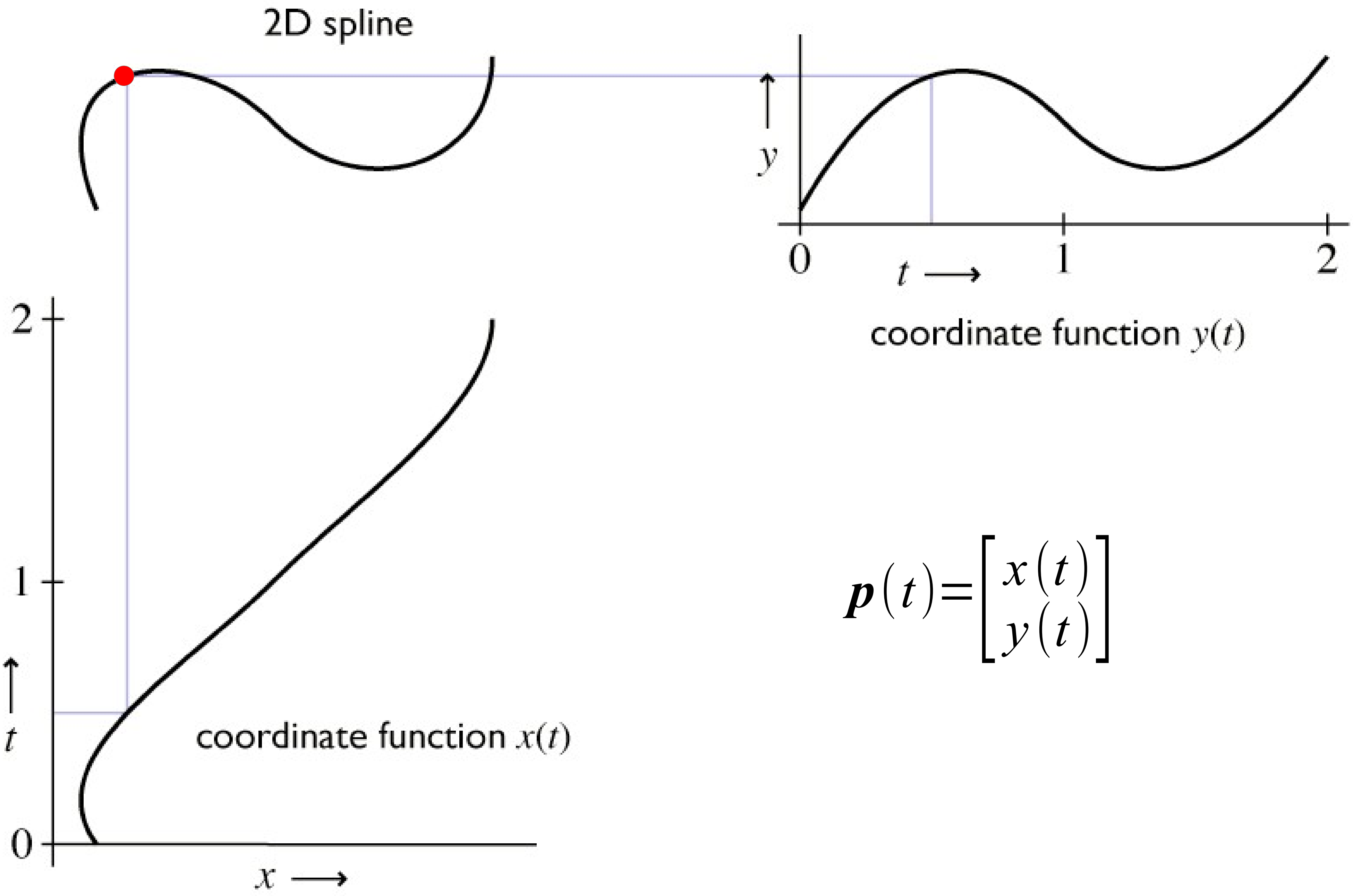
$$\boldsymbol{p}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

Coordinate Functions



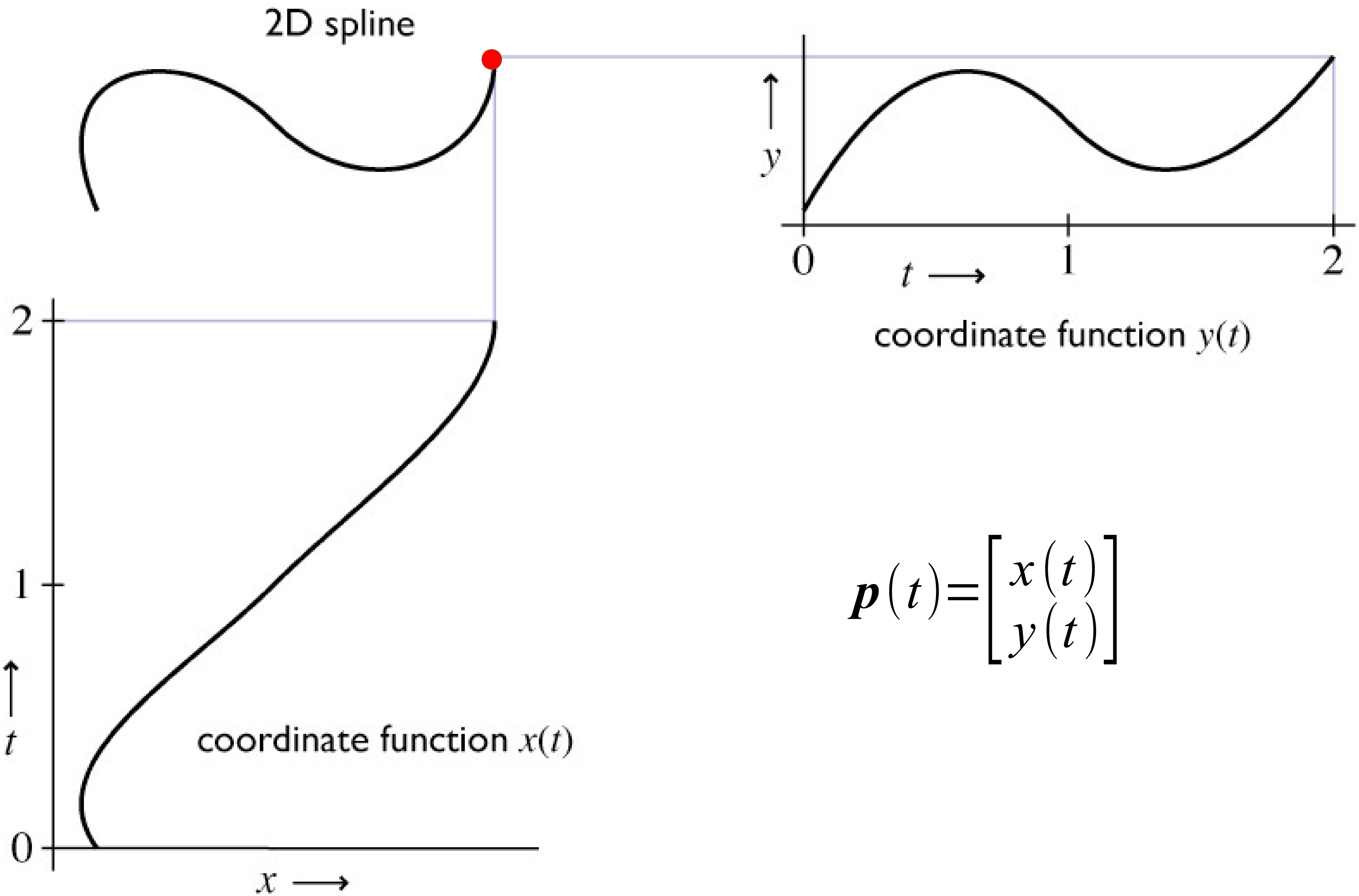
$$\boldsymbol{p}(t)=\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

Coordinate Functions



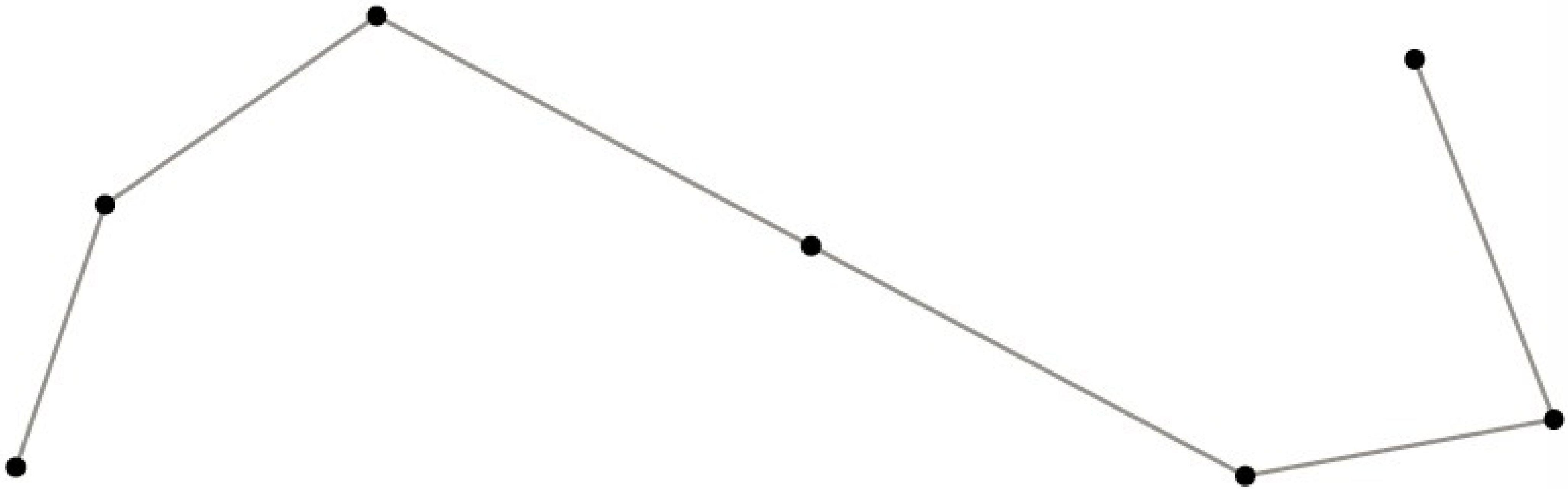
$$\boldsymbol{p}(t)=\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

Coordinate Functions



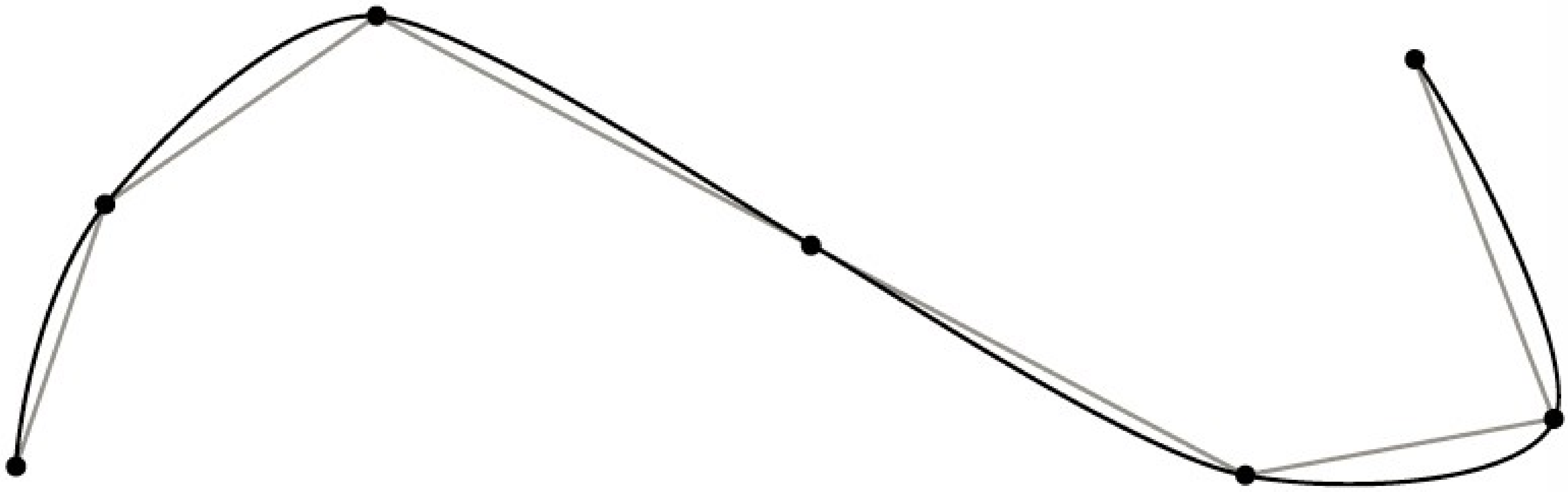
$$\boldsymbol{p}(t)=\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

Splines



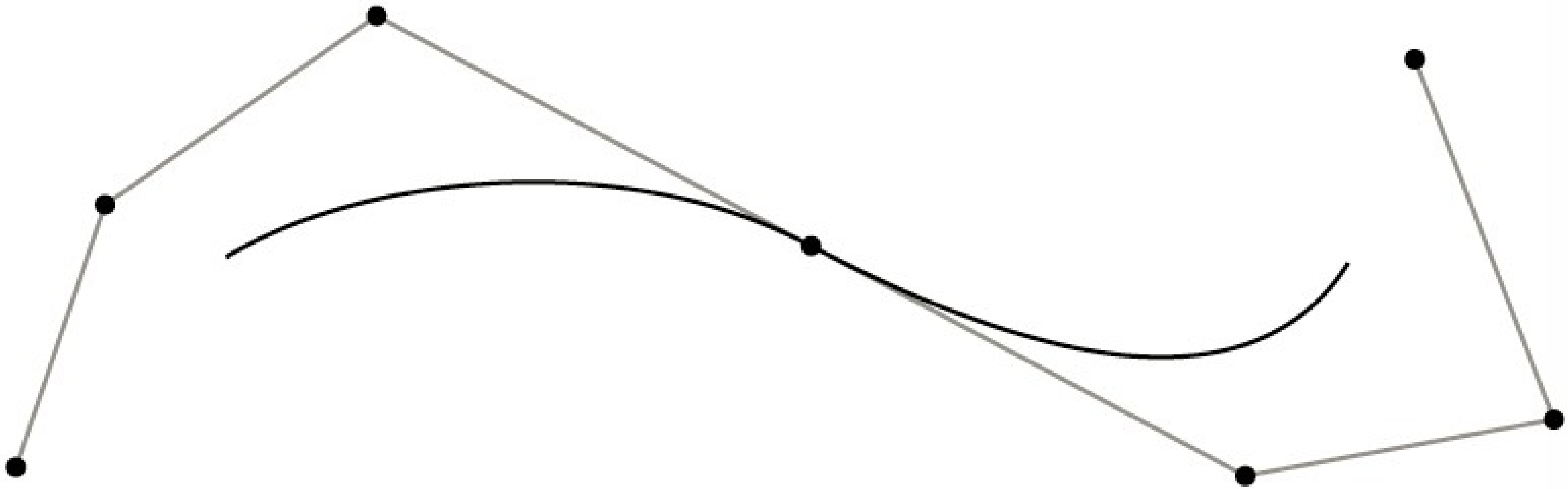
Control Points: Control the shape of the spline

Splines



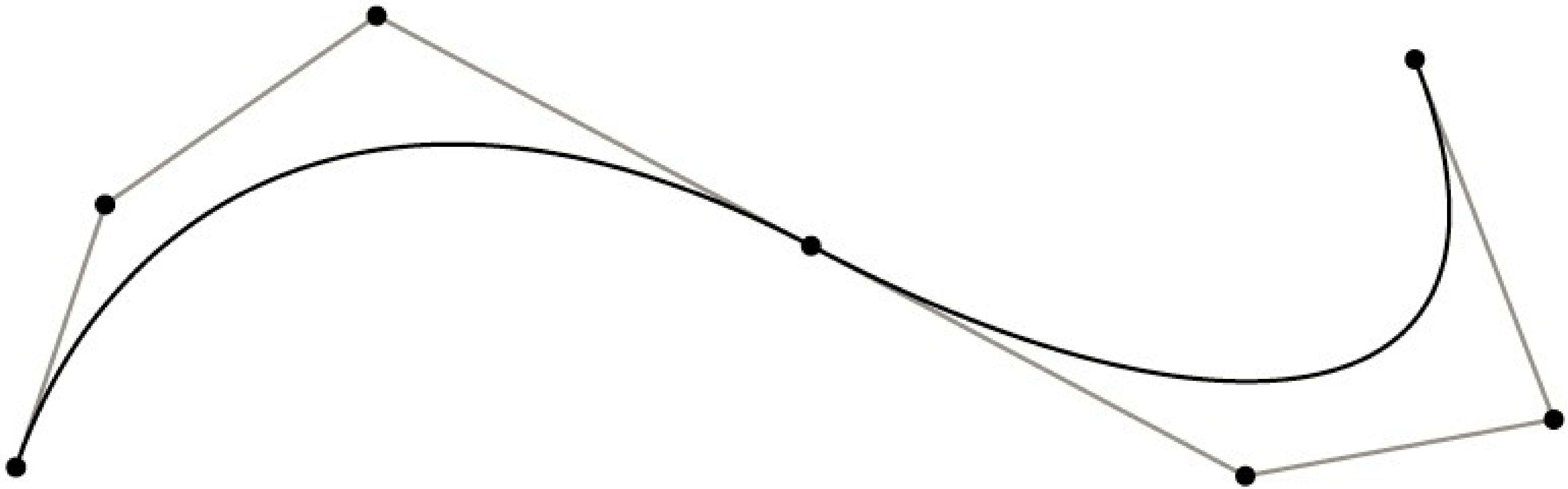
Interpolating spline: passes through the control points

Splines



Approximating Spline: just guided by the control points

Splines



Mixture: goes through some and approximates some

Splines

- Each coordinate function is treated separately

$$\boldsymbol{p}(t) = [x(t) \quad y(t)]$$

- Two Formulations:
  - Polynomial in t (Polynomial Formulation)

$$\boldsymbol{p}(t) = \sum_i t^i \boldsymbol{a}_{\boldsymbol{i}}$$

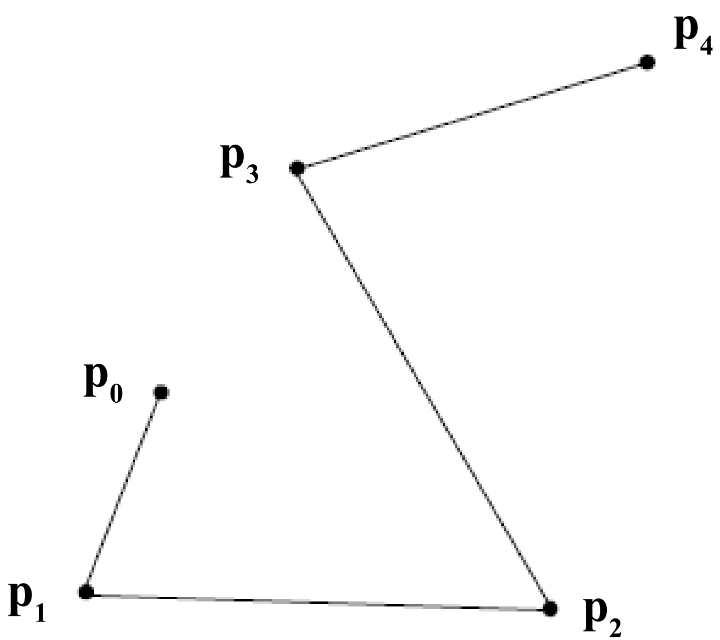
  - Linear combinations of the control points (Basis Function Formulation)

$$\boldsymbol{p}(t) = \sum_i b_i(t) \boldsymbol{p}_{\boldsymbol{i}}$$

Linear Splines

Represent each interval as a straight line

$$x(t) = x_0 + (x_1 - x_0)t$$
$$y(t) = y_0 + (y_1 - y_0)t$$
$$\boldsymbol{p}(t) = \boldsymbol{p}_0 + (\boldsymbol{p}_1 - \boldsymbol{p}_0)t$$



Linear Splines

Polynomial Formulation: $\boldsymbol{p}(t) = \boldsymbol{a}_0 + t\,\boldsymbol{a}_1$

Constraints:

$$\boldsymbol{p}(0) = \boldsymbol{p}_0 = \boldsymbol{a}_0$$
$$\boldsymbol{p}(1) = \boldsymbol{p}_1 = \boldsymbol{a}_0 + \boldsymbol{a}_1$$

Matrix Form:

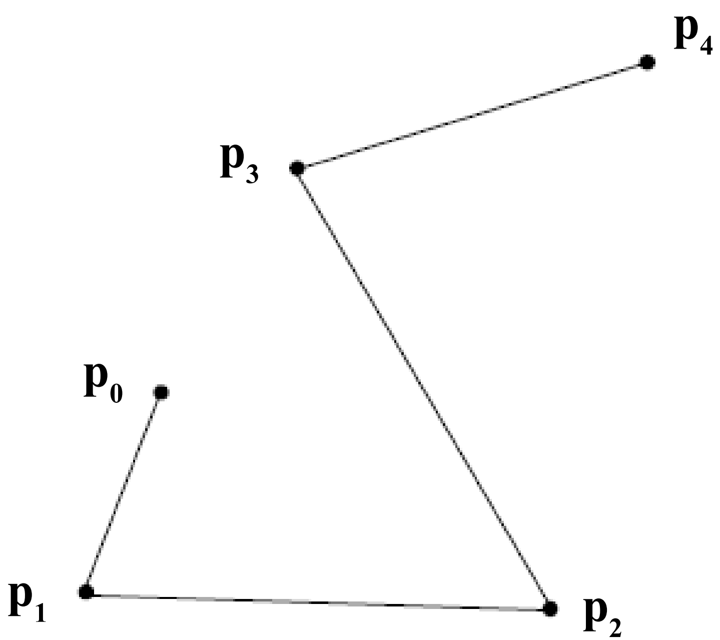
$$\begin{bmatrix} \boldsymbol{p}_0 \\ \boldsymbol{p}_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_0 \end{bmatrix}$$

Constraint Matrix: $\boldsymbol{p} = C\,\boldsymbol{a}$

Solve for $\boldsymbol{a}$: $\boldsymbol{a} = B\,\boldsymbol{p}$

Basis Matrix: $B = C^{-1}$

$$B = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$$



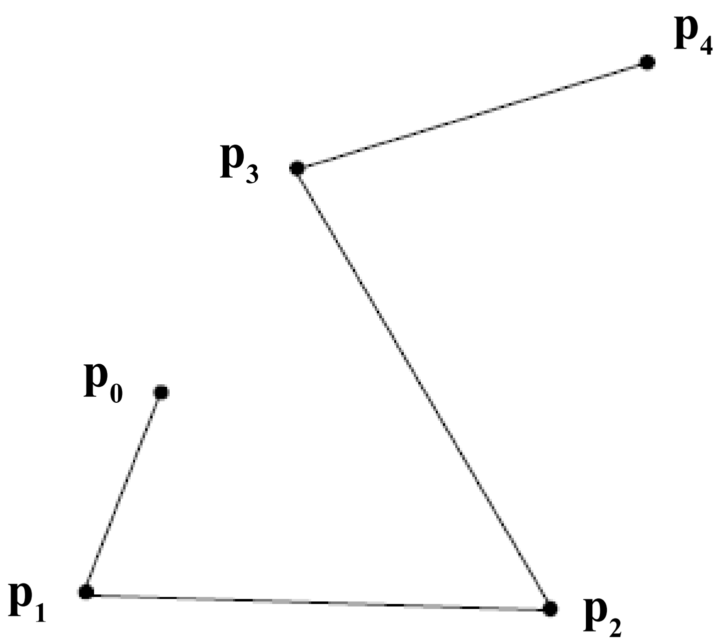
Linear Splines

Polynomial Formulation: $\boldsymbol{p}(t)=\boldsymbol{a}_0+t\,\boldsymbol{a}_1$

$$\begin{bmatrix}\boldsymbol{a}_1\\\boldsymbol{a}_0\end{bmatrix}=\begin{bmatrix}-1 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{p}_0\\\boldsymbol{p}_1\end{bmatrix}$$

$$\boldsymbol{a}_0=\boldsymbol{p}_0$$
$$\boldsymbol{a}_1=\boldsymbol{p}_1-\boldsymbol{p}_0$$

$$\boldsymbol{a}=B\,\boldsymbol{p}$$



Linear Splines

Matrix Form

$$\boldsymbol{p}(t) = \boldsymbol{a}_0 + t\,\boldsymbol{a}_1$$

$$\boldsymbol{p}(t) = \begin{bmatrix} t & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_0 \end{bmatrix}$$

$$\begin{bmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_0 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{p}_0 \\ \boldsymbol{p}_1 \end{bmatrix}$$

$$\boldsymbol{p}(t) = \begin{bmatrix} t & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{p}_0 \\ \boldsymbol{p}_1 \end{bmatrix}$$

$$\boldsymbol{p}(t) = \boldsymbol{t}\,B\,\boldsymbol{p}$$

$$\boldsymbol{t} = \begin{bmatrix} t & 1 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \qquad \boldsymbol{p} = \begin{bmatrix} \boldsymbol{p}_0 \\ \boldsymbol{p}_1 \end{bmatrix}$$

Linear Splines

Polynomial Formulation: $\boldsymbol{p}(t)=\boldsymbol{a}_0+t\,\boldsymbol{a}_1$

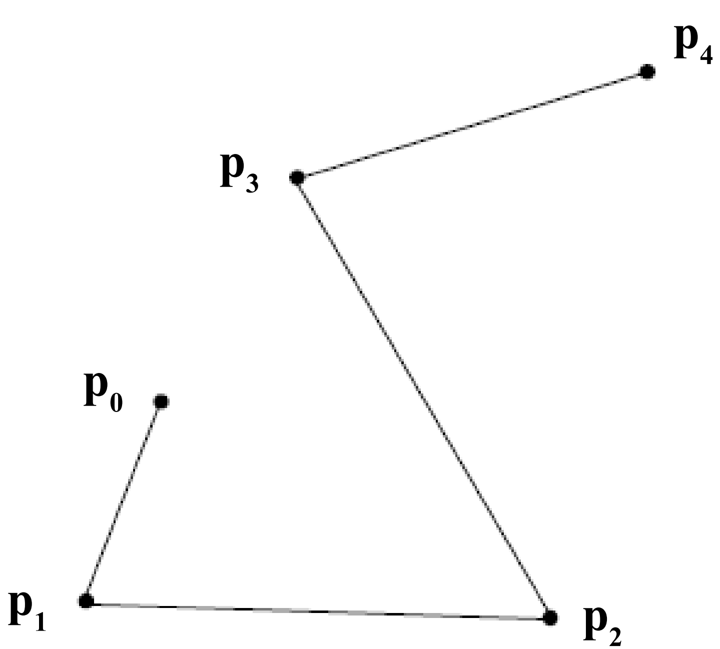
$$\boldsymbol{p}(t)=\boldsymbol{p}_0+t(\boldsymbol{p}_1-\boldsymbol{p}_0)$$

Matrix Form

$$\boldsymbol{p}(t)=\begin{bmatrix} t & 1 \end{bmatrix}\left(\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{p}_0 \\ \boldsymbol{p}_1 \end{bmatrix}\right)$$

$$\boldsymbol{p}(t)=\boldsymbol{t}(B\,\boldsymbol{p})$$

$$\boldsymbol{t}=\begin{bmatrix} t & 1 \end{bmatrix} \qquad B=\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \qquad \boldsymbol{p}=\begin{bmatrix} \boldsymbol{p}_0 \\ \boldsymbol{p}_1 \end{bmatrix}$$



Linear Splines

Basis Function Formulation $\boldsymbol{p}(t) = \sum_i b_i(t)\, \boldsymbol{p_i}$

$$\boldsymbol{p}(t) = (\boldsymbol{p}_1 - \boldsymbol{p}_0)\, t + \boldsymbol{p}_0$$

$$= (1 - t)\, \boldsymbol{p}_0 + t\, \boldsymbol{p}_1$$

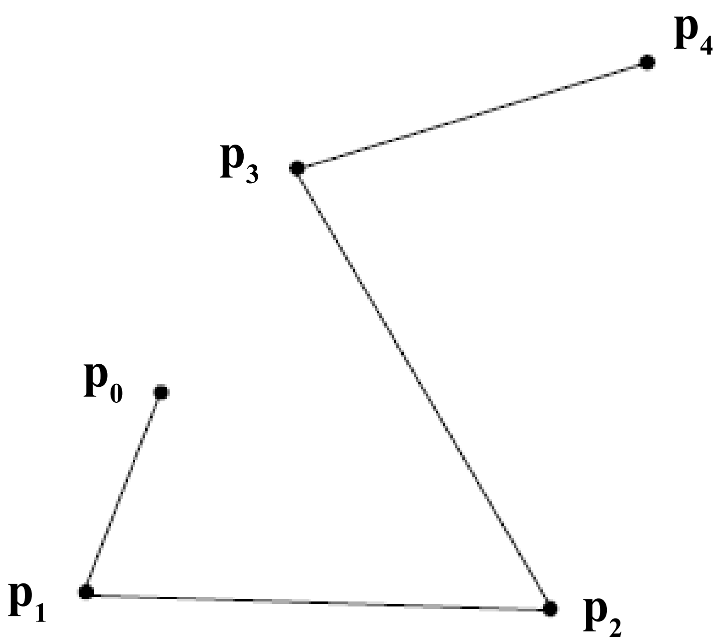
Matrix Form

$$\boldsymbol{p}(t) = \left(\begin{bmatrix} t & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \right) \begin{bmatrix} \boldsymbol{p}_0 \\ \boldsymbol{p}_1 \end{bmatrix}$$

$$\boldsymbol{p}(t) = (\boldsymbol{t}\, B)\, \boldsymbol{p}$$

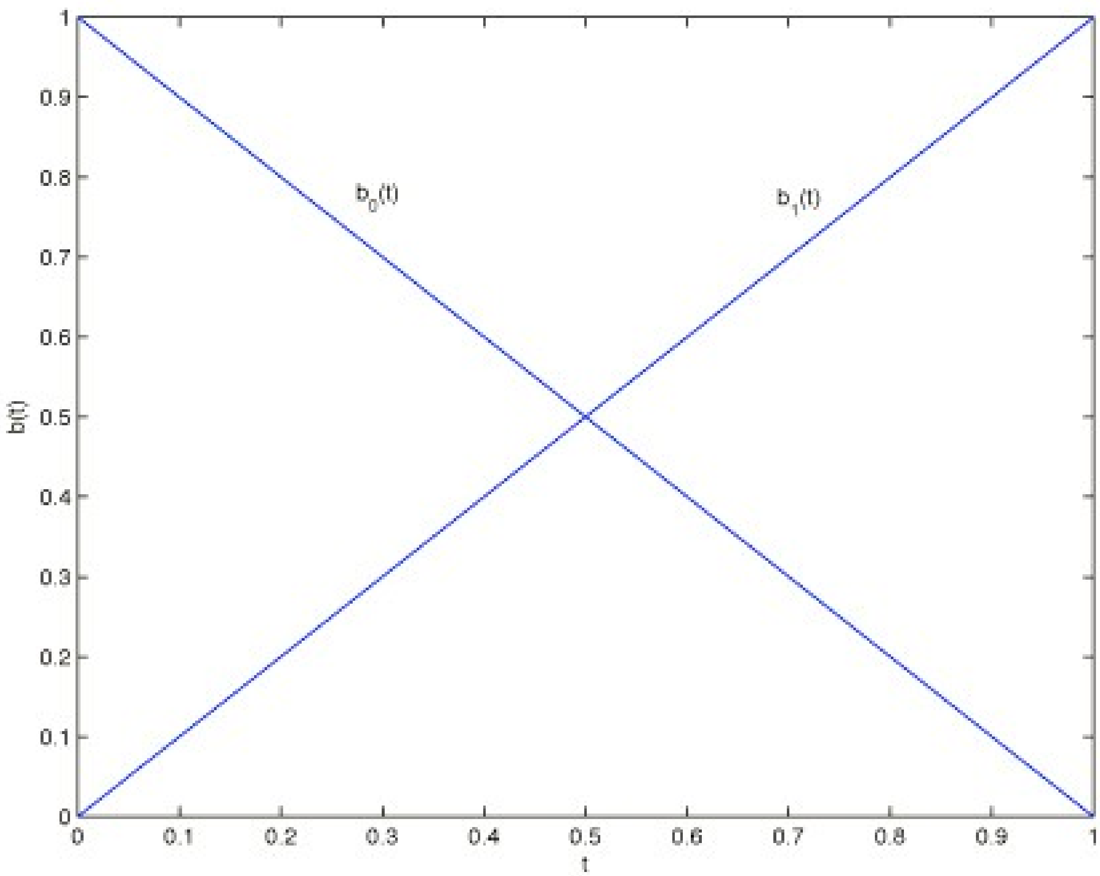
Basis Functions: $\boldsymbol{b}(t) = \boldsymbol{t}\, B$

$$\boldsymbol{b}(t) = \begin{bmatrix} \boldsymbol{b}_0(t) \\ \boldsymbol{b}_1(t) \end{bmatrix} = \begin{bmatrix} 1 - t \\ t \end{bmatrix}$$



Linear Splines

$$\boldsymbol{p}(t) = (1-t)\,\boldsymbol{p_0} + t\,\boldsymbol{p_1} = b_0(t)\,\boldsymbol{p_0} + b_1(t)\,\boldsymbol{p_1}$$

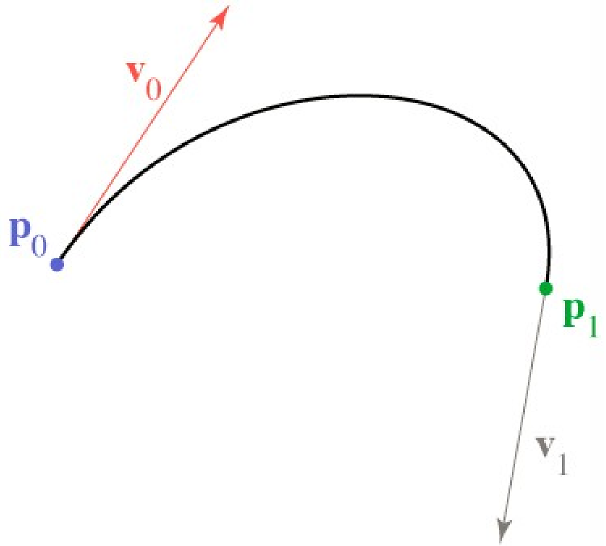


Blending (Basis) Functions: Contribution of each point as t changes

Hermite Splines

- Piecewise cubic polynomials
- Constraints:
  - two end points
  - two tangents (derivatives)

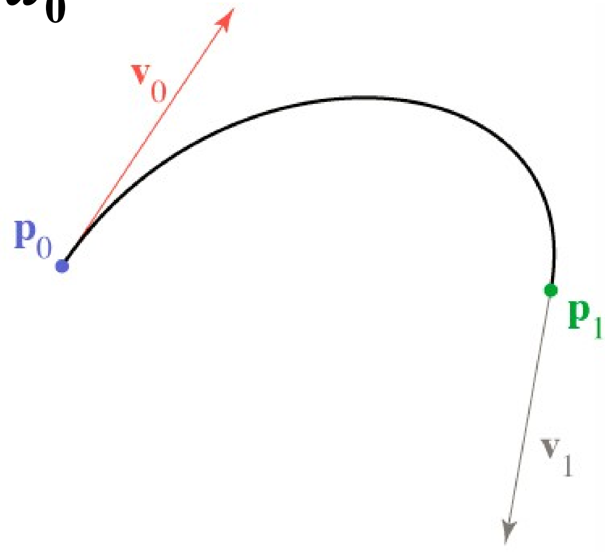
$$\boldsymbol{p}(t) = \boldsymbol{a}_3 t^3 + \boldsymbol{a}_2 t^2 + \boldsymbol{a}_1 t + \boldsymbol{a}_0$$



Hermite Splines

$$p(t) = a_3 t^3 + a_2 t^2 + a_1 t + a_0$$

$$p'(t) = 3 a_3 t^2 + 2 a_2 t + a_1$$



Constraints

$$p(0) = p_0 = a_0$$

$$p(1) = p_1 = a_3 + a_2 + a_1 + a_0$$

$$p'(0) = v_0 = a_1$$

$$p'(1) = v_1 = 3 a_3 + 2 a_2 + a_1$$

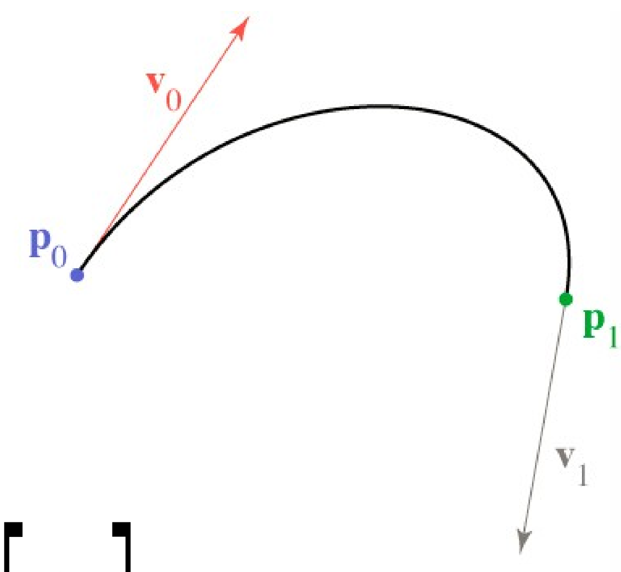
$$\begin{bmatrix} p_0 \\ p_1 \\ v_0 \\ v_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix}$$

Solve for $\boldsymbol{a} = (a_3, a_2, a_1, a_0)$

Hermite Splines

$$\begin{bmatrix} p_0 \\ p_1 \\ v_0 \\ v_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix}$$



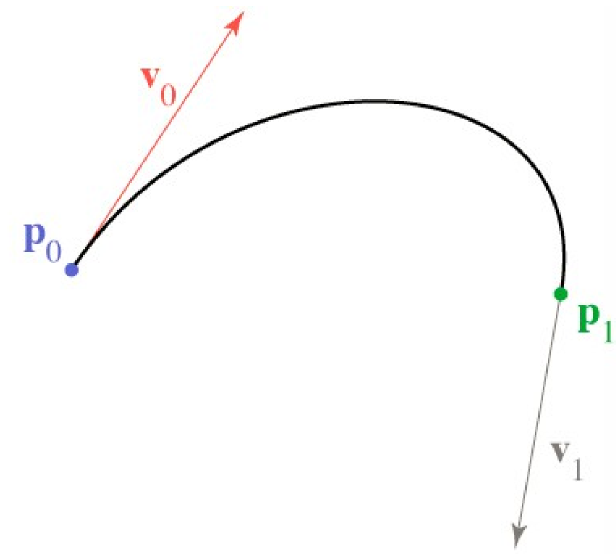
$$a = C^{-1} p = B\, p$$

$$\begin{bmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ v_0 \\ v_1 \end{bmatrix}$$

$$a_3 = 2\, p_0 - 2\, p_1 + v_0 + v_1$$
$$a_2 = -3\, p_0 + 3\, p_1 - 2\, v_0 - v_1$$
$$a_1 = v_0$$
$$a_0 = p_0$$

Hermite Splines

$$\begin{bmatrix} \boldsymbol{a}_3 \\ \boldsymbol{a}_2 \\ \boldsymbol{a}_1 \\ \boldsymbol{a}_0 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{p}_0 \\ \boldsymbol{p}_1 \\ \boldsymbol{v}_0 \\ \boldsymbol{v}_1 \end{bmatrix}$$



$$\boldsymbol{p}(t) = \boldsymbol{t}\,\boldsymbol{a} = \boldsymbol{t}\,B\,\boldsymbol{p}$$

$$\boldsymbol{p}(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{p}_0 \\ \boldsymbol{p}_1 \\ \boldsymbol{v}_0 \\ \boldsymbol{v}_1 \end{bmatrix}$$

Hermite Splines

Basis Function Formulation $\boldsymbol{p}(t)=\sum_i b_i(t)\,\boldsymbol{p_i}$

$$\boldsymbol{p}(t)=\begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix}\begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{p_0} \\ \boldsymbol{p_1} \\ \boldsymbol{v_0} \\ \boldsymbol{v_1} \end{bmatrix}$$

$$\boldsymbol{b}(t)=\boldsymbol{t}\,B$$

Hermite Polynomials

$$b_0(t)=2t^3-3t^2+1$$

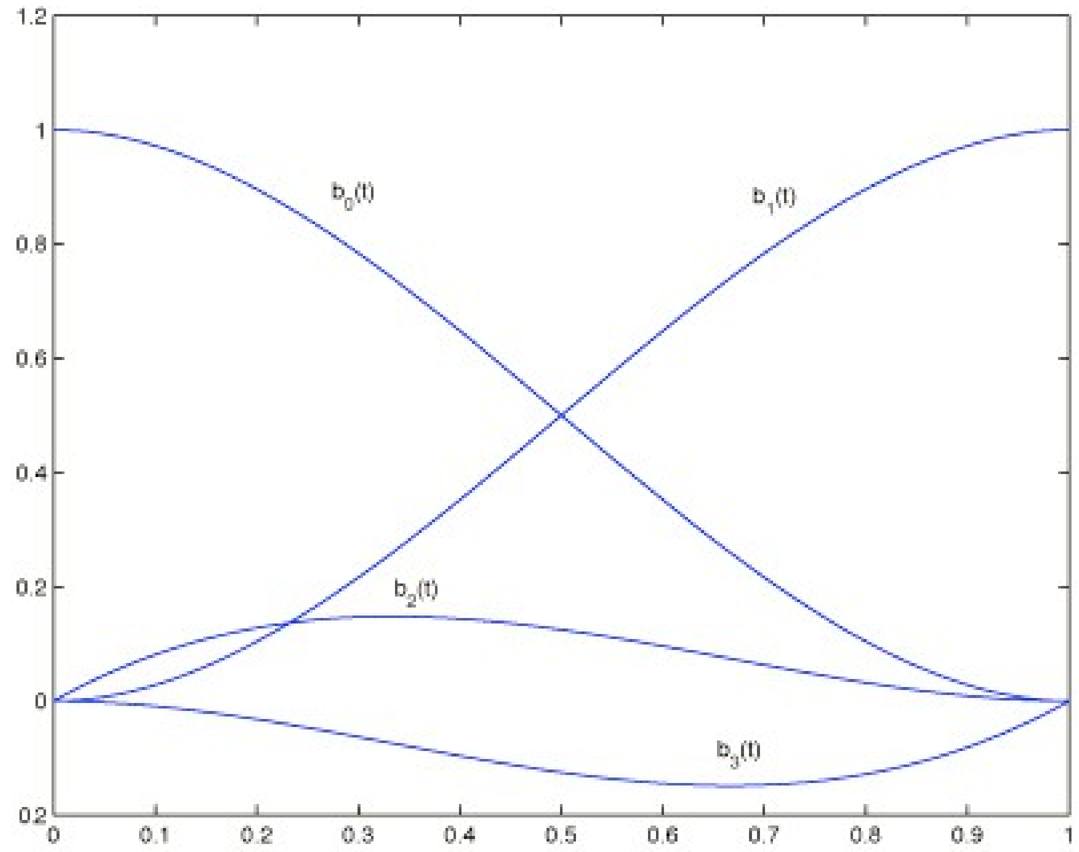
$$b_1(t)=-2t^3+3t^2$$

$$b_2(t)=t^3-2t^2+t$$

$$b_3(t)=t^3-t^2$$

Hermite Splines

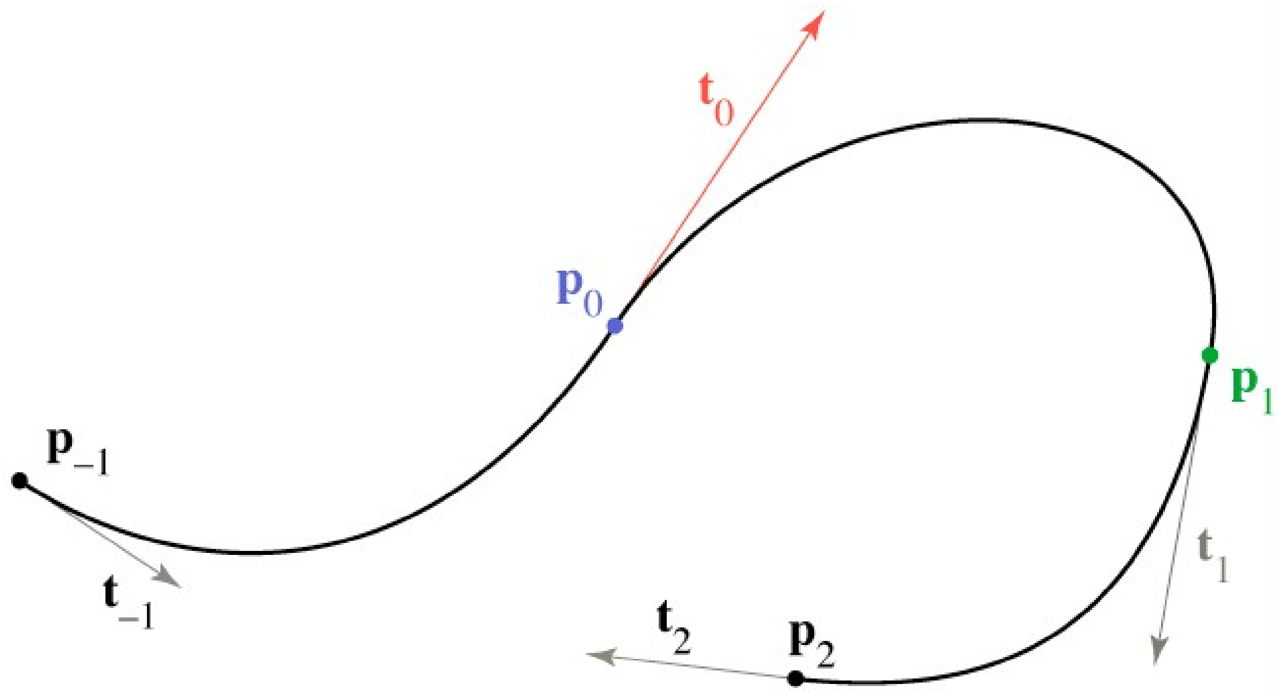
Basis Function Formulation $\boldsymbol{p}(t) = \sum_i b_i(t)\,\boldsymbol{p}_i$



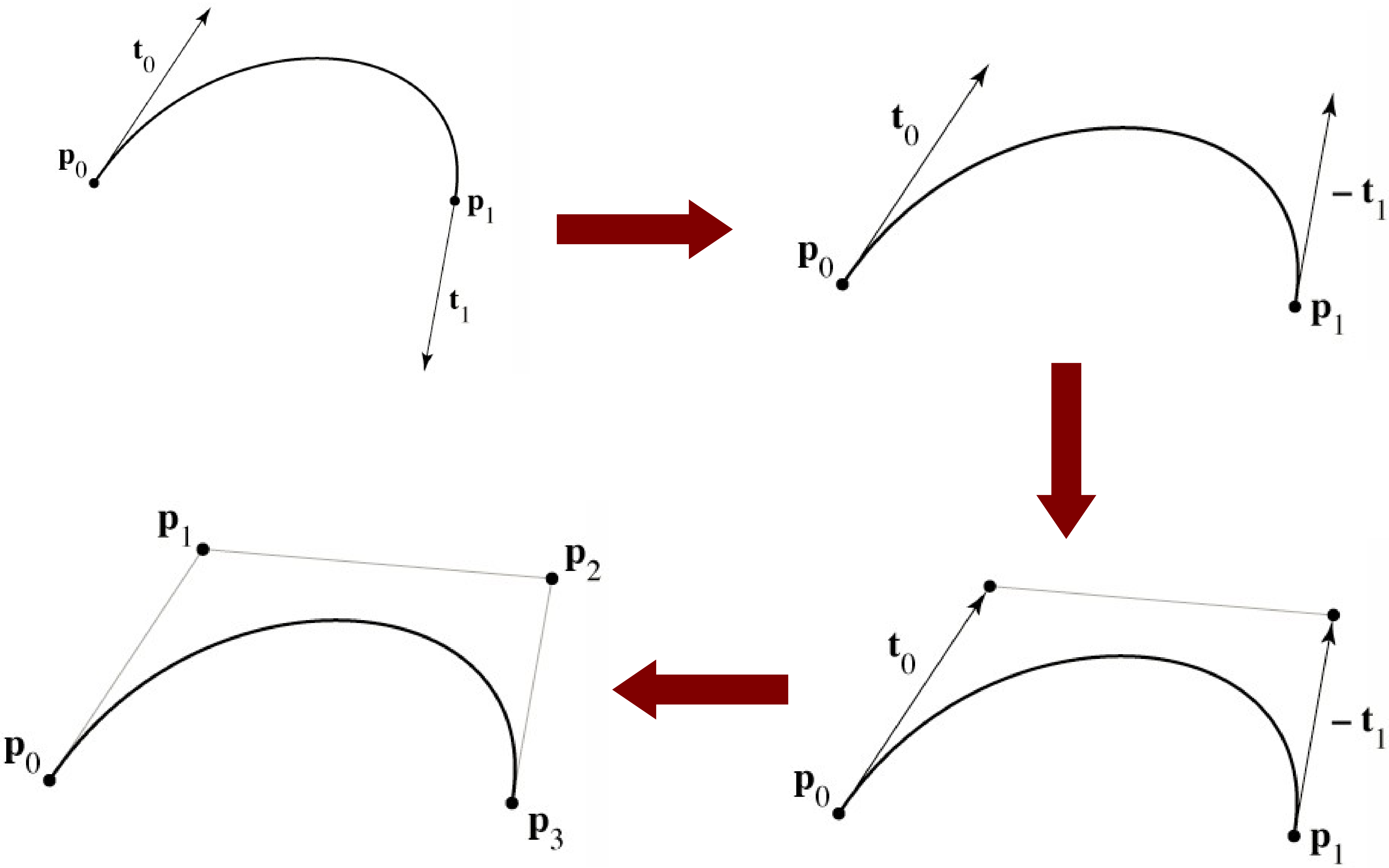
Blending Functions

Hermite Splines

- Longer splines
  - Split into pieces
  - Join pieces such that tangents match



Bézier Splines



Represent tangents as difference between points

Bézier Splines

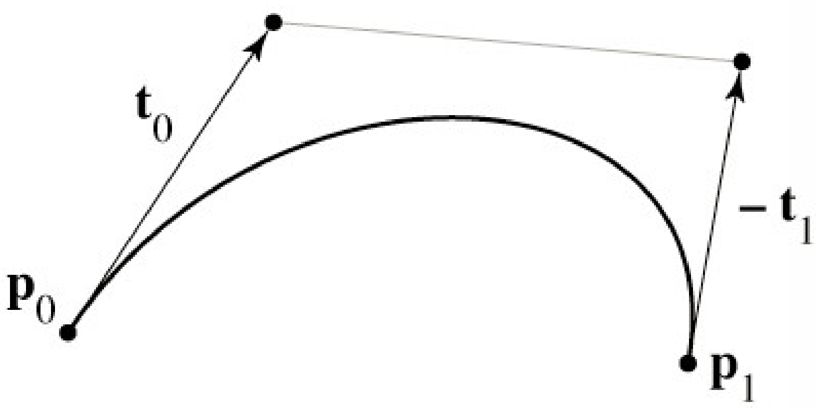
$$p(t)=a_3 t^3+a_2 t^2+a_1 t+a_0$$

$$p'(t)=3a_3 t^2+2a_2 t+a_1$$

Constraints

$$p(0)=p_0=a_0$$
$$p(1)=p_3=a_3+a_2+a_1+a_0$$
$$p'(0)=3(p_1-p_0)=a_1$$
$$p'(1)=3(p_3-p_2)=3a_3+2a_2+a_1$$

$$\begin{bmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix}=\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}$$



Can also derive from Hermite Splines. How?

Bézier Splines

$$\boldsymbol{p}(t) = \boldsymbol{a}_3 t^3 + \boldsymbol{a}_2 t^2 + \boldsymbol{a}_1 t + \boldsymbol{a}_0$$

$$\begin{bmatrix} \boldsymbol{a}_3 \\ \boldsymbol{a}_2 \\ \boldsymbol{a}_1 \\ \boldsymbol{a}_0 \end{bmatrix} = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{p}_0 \\ \boldsymbol{p}_1 \\ \boldsymbol{p}_2 \\ \boldsymbol{p}_3 \end{bmatrix}$$

$$\boldsymbol{p}(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{p}_0 \\ \boldsymbol{p}_1 \\ \boldsymbol{p}_2 \\ \boldsymbol{p}_3 \end{bmatrix}$$

Bézier Splines

$$\boldsymbol{p}(t)=\boldsymbol{b}_0(t)\,\boldsymbol{p}_0+\boldsymbol{b}_1(t)\,\boldsymbol{p}_1+\boldsymbol{b}_2(t)\,\boldsymbol{p}_2+\boldsymbol{b}_3(t)\,\boldsymbol{p}_3$$

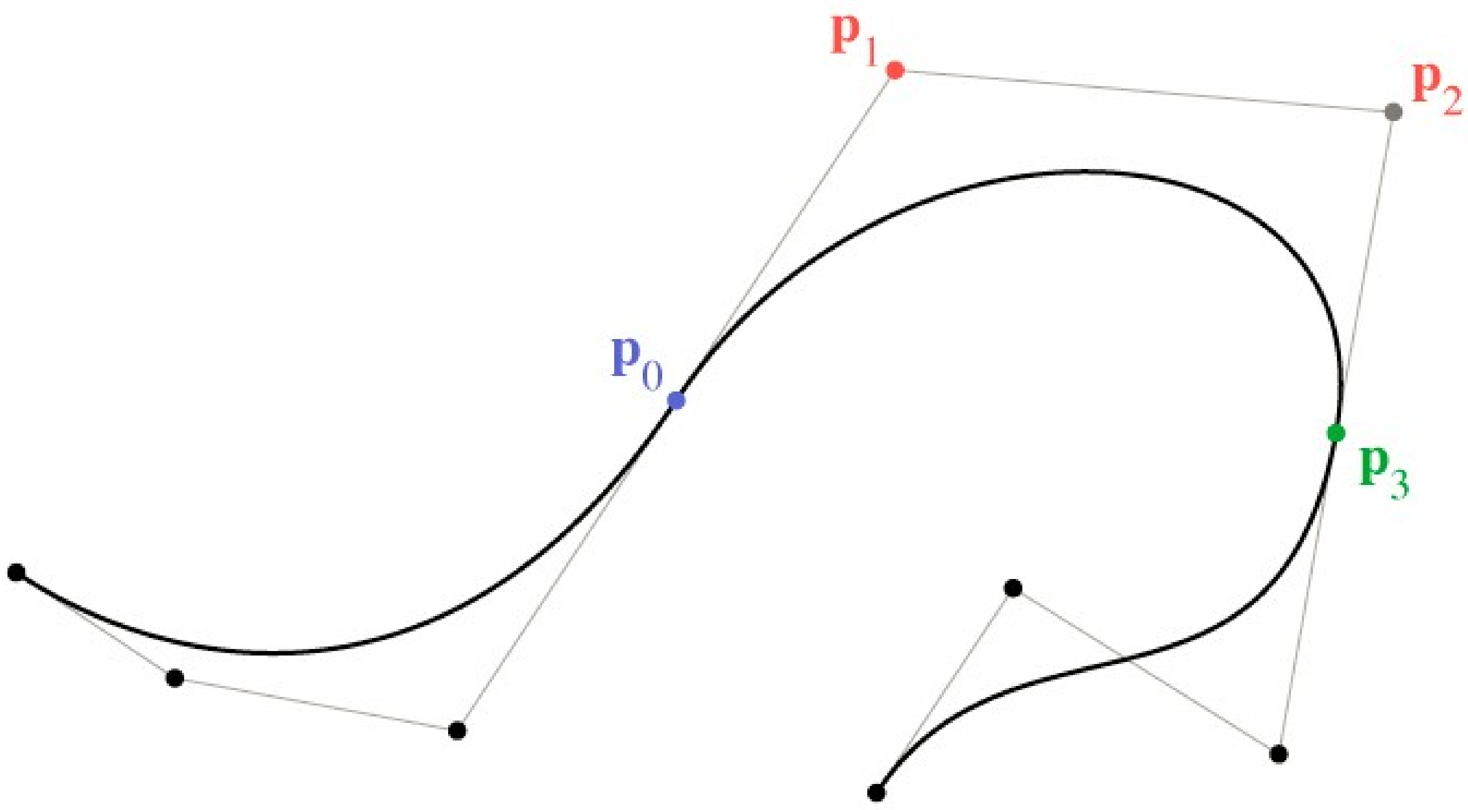
$$\boldsymbol{p}(t)=\left(\begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix}\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\right)\begin{bmatrix} \boldsymbol{p}_0 \\ \boldsymbol{p}_1 \\ \boldsymbol{p}_2 \\ \boldsymbol{p}_3 \end{bmatrix}$$

Basis Functions

$$\boldsymbol{b}(t)=\begin{bmatrix} -t^3+3t^2-3t+1 \\ 3t^3-6t^2+3t \\ -3t^3+3t^2 \\ t^3 \end{bmatrix}=\begin{bmatrix} (1-t)^3 \\ 3(1-t)^2 t \\ 3(1-t)t^2 \\ t^3 \end{bmatrix}$$

Bernstein Polynomials

Bézier Splines



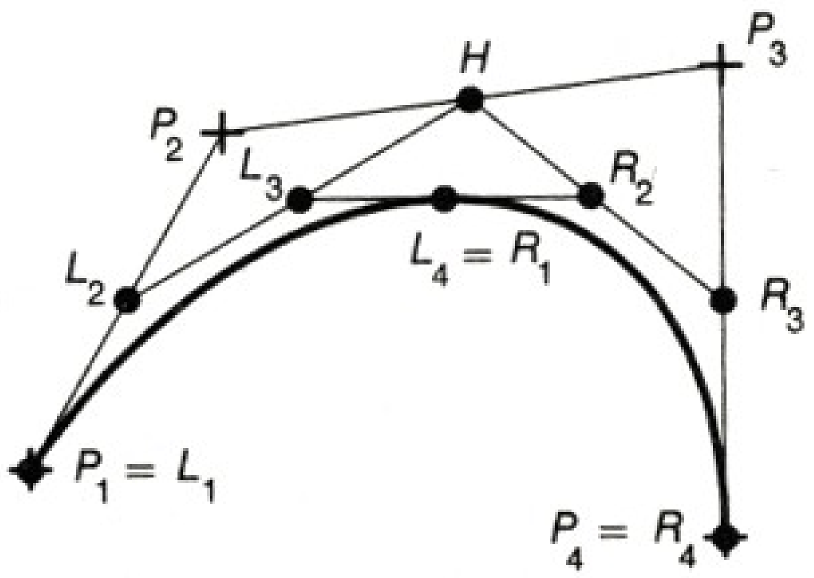
Chaining Bézier Splines: Collinear control points

Splines

- Other types:
  - Catmull-Rom
  - B-splines
  - Non-Uniform B-splines
  - Non-Uniform Rational B-splines (NURBs)

Spline Drawing

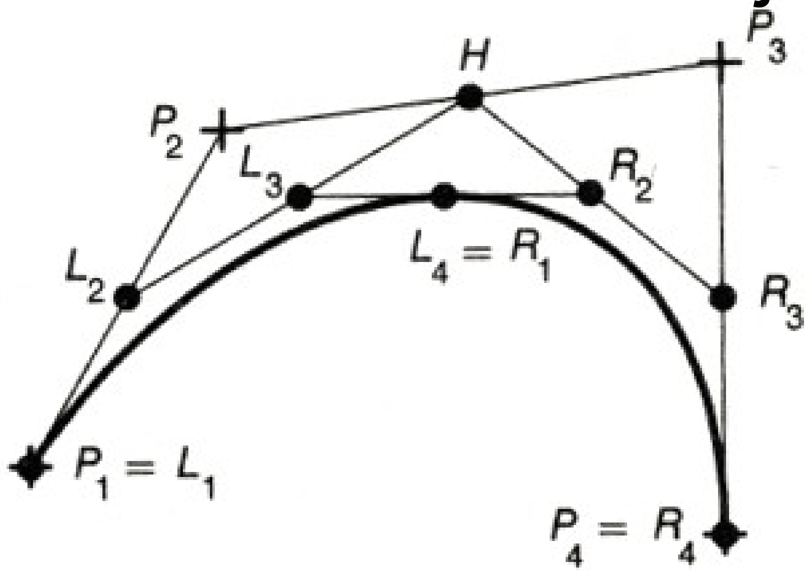
$$\boldsymbol{p}(t) = \boldsymbol{t}\, B\, \boldsymbol{p}$$



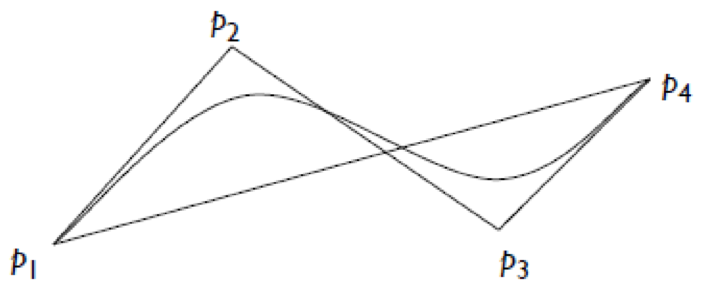
Need to rasterize the the spline to display it

Spline Drawing

- Recursive Subdivision: De Casteljau's Algorithm



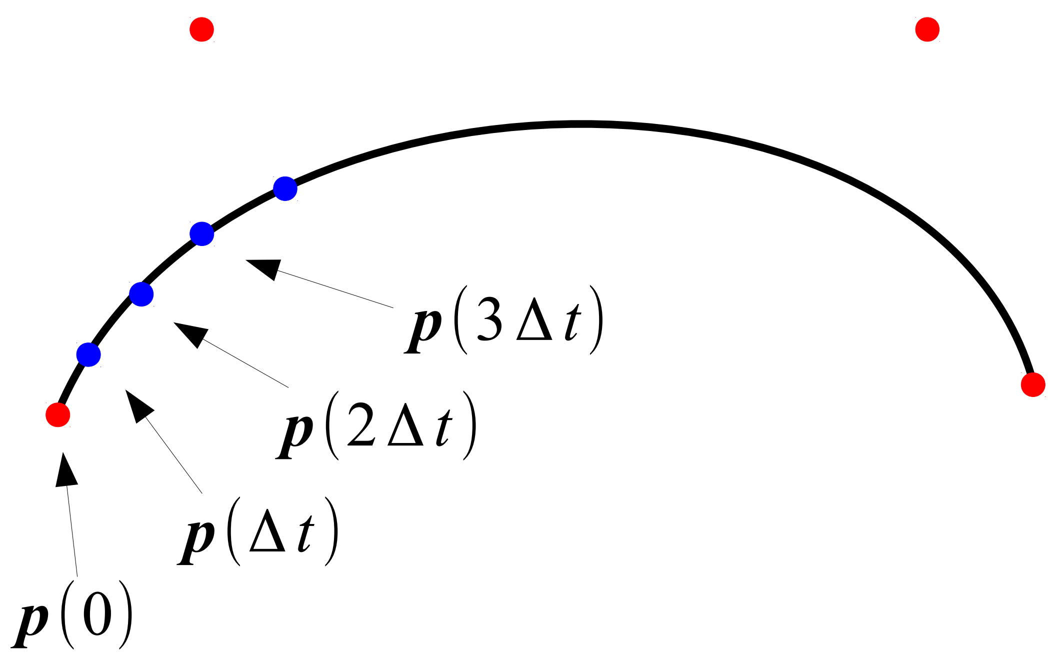
- Termination
  - Distance between control points and line
  - Distance between control points



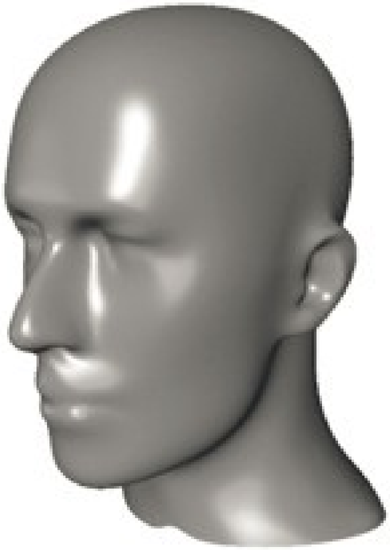
Spline Drawing

- Uniform Spacing
  - Subdivide the range of t using a fixed step Δt
  - Incrementally compute points $p(t + \Delta t)$ from $p(t)$

$$\boldsymbol{p}(t+\Delta t)=\boldsymbol{p}(t)+\Delta \boldsymbol{p}(\boldsymbol{p}(t))$$

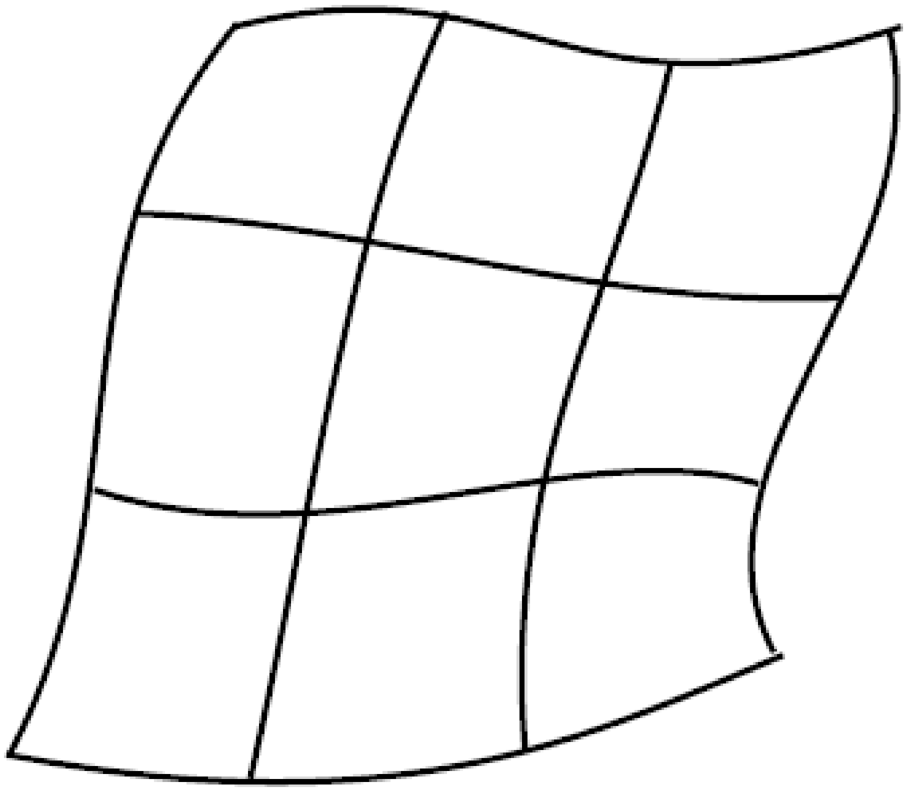


Surfaces



Surfaces used for modeling

Surfaces



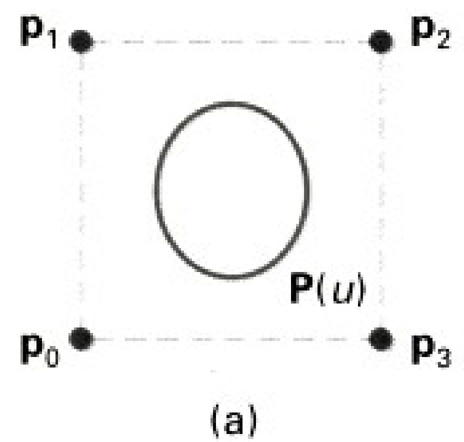
How many dimensions? 2D curve in 3D space

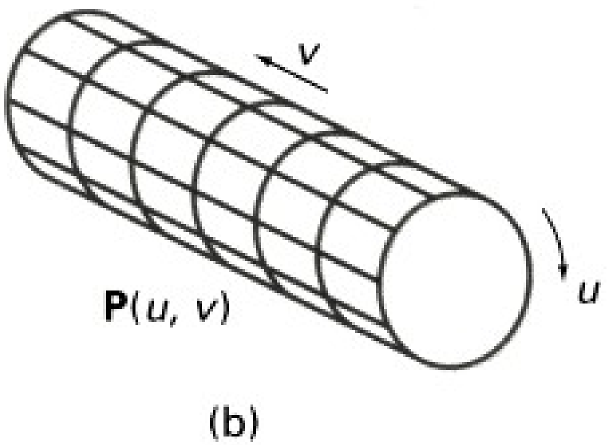
Surfaces

- Representing surfaces
  - Extrusions
  - Surfaces of Revolution
  - Swept Surfaces
  - Spline Patches
  - Subdivision Surface

Extrusions

Cylinder





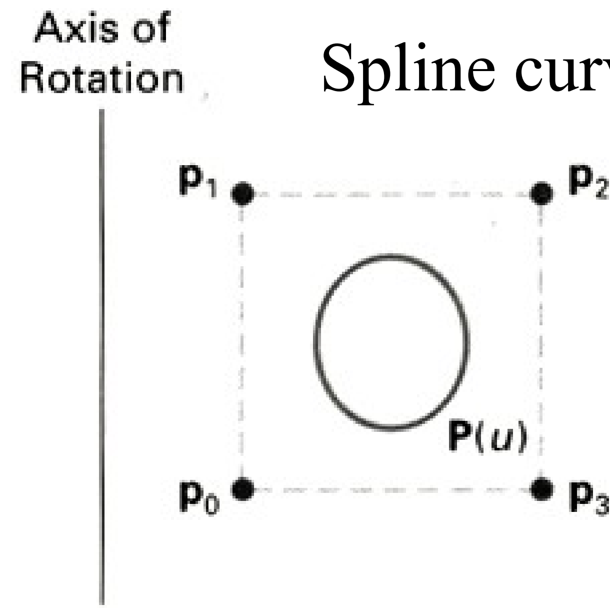
Spline curve: $P(u)$

Surface: $P(u, v)$

Slide the curve *P(u)* along axis

Surfaces of Revolution

Spline curve: $P(u)$

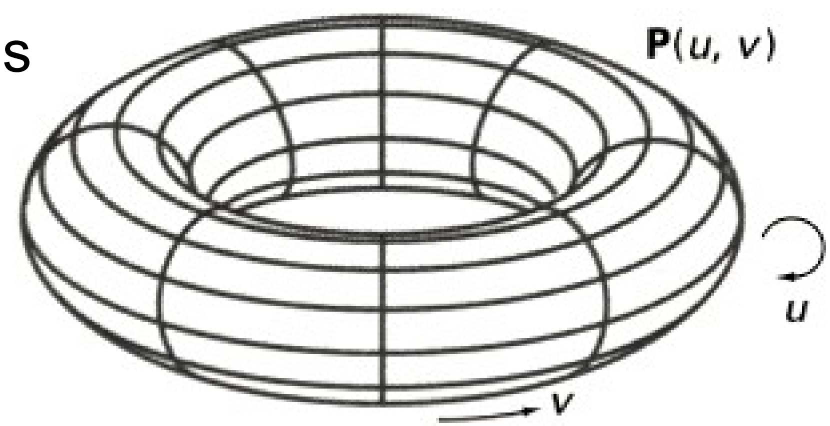


(a)

Revolve the curve *P(u)* around axis

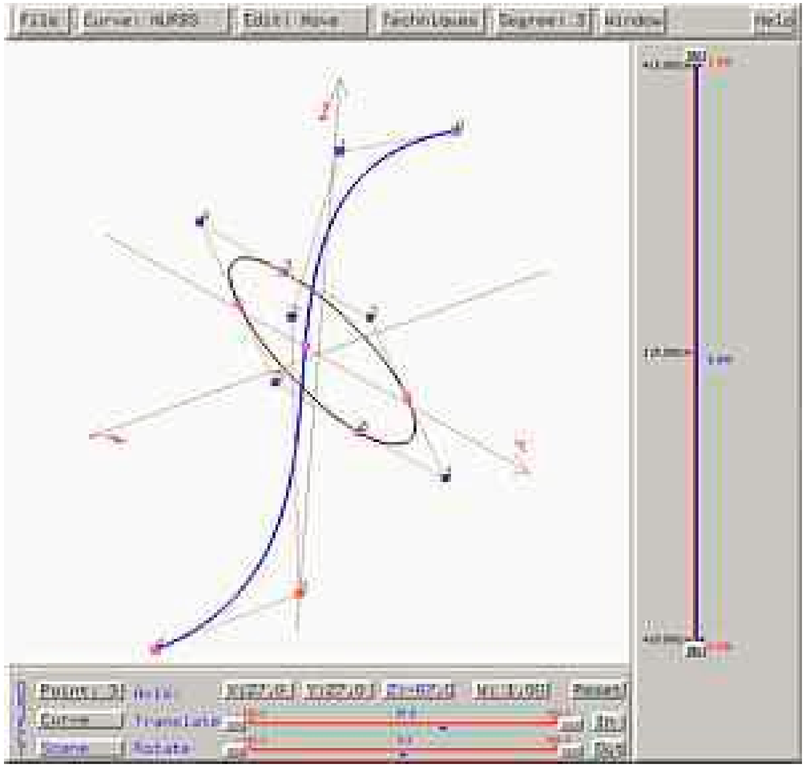
Surface: $P(u, v)$

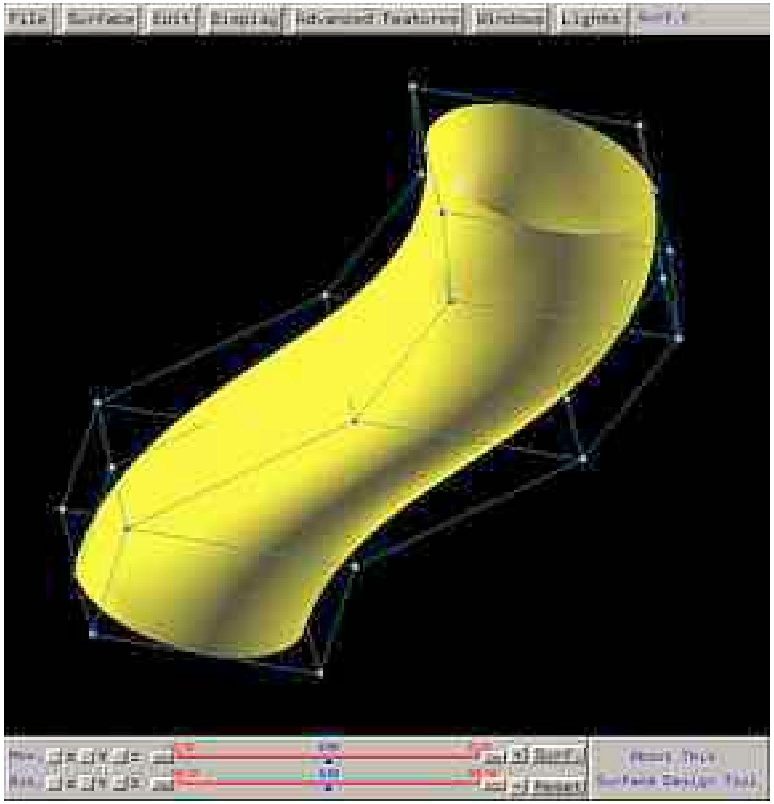
Torus



(b)

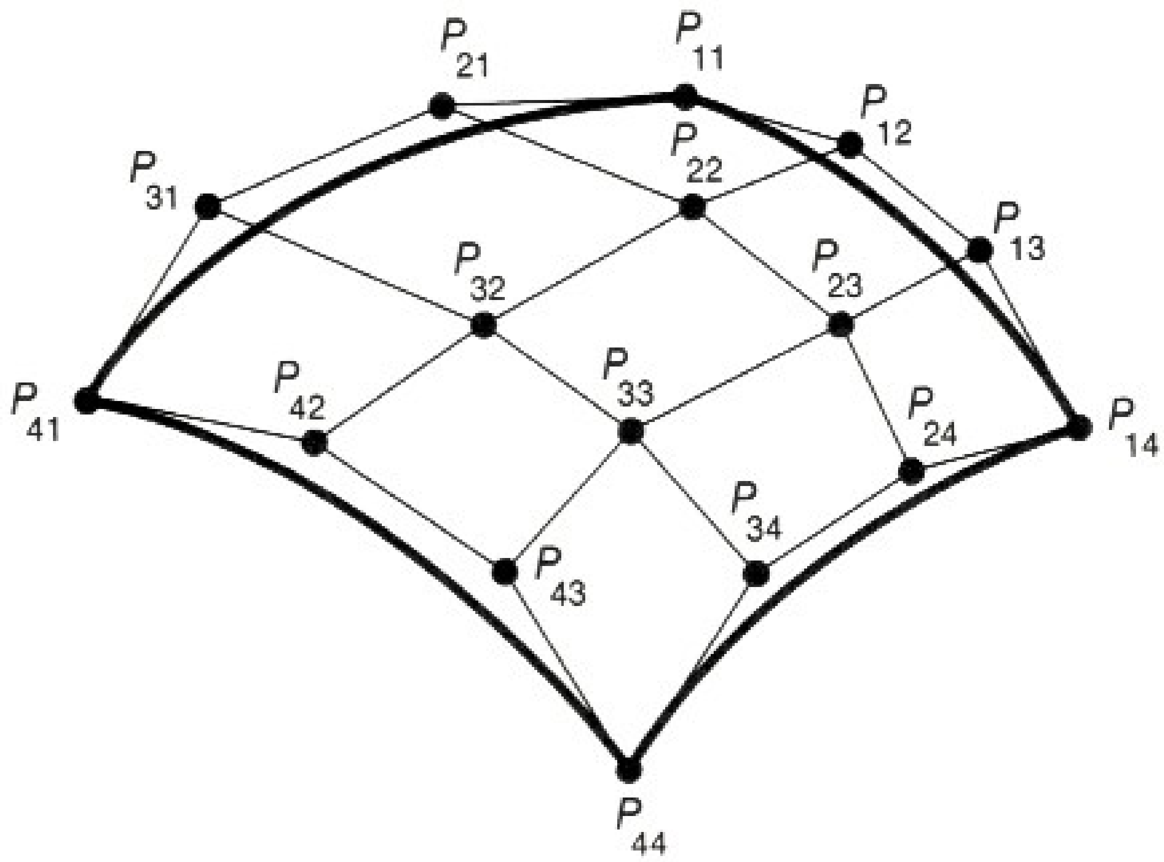
Swept Surfaces





Sweep a *cross section* along a *spine*

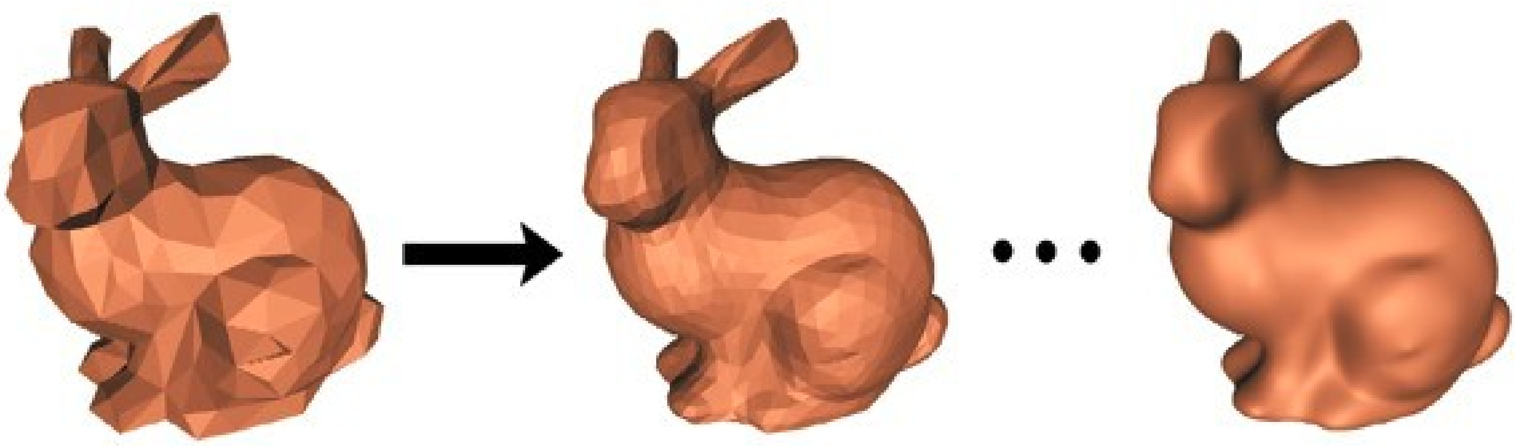
Bicubic Bézier Patches



16 control points

Subdivision Surfaces

- Start with polygonal mesh
- Subdivide into larger number of polygons
- Results in smoother surfaces



Recap

- Splines
  - Linear Splines
  - Hermite Splines
  - Bézier Splines
- Surfaces